

# Quasi-Metrizability of Bornological Biuniverses in $\mathbf{ZF}$

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## Abstract

Hu's metrization theorem for bornological universes is shown to hold in  $\mathbf{ZF}$  and it is adapted to a quasi-metrization theorem for bornologies in bitopological spaces. The problem of uniform quasi-metrization of quasi-metric bornological universes is investigated. Several consequences for natural bornologies in generalized topological spaces in the sense of Delfs and Knebusch are deduced. Some statements concerning (uniform)-(quasi)-metrization of bornologies are shown to be relatively independent of  $\mathbf{ZF}$ .

# 1 Introduction

A **bitopological space** is a triple  $(X, \tau_1, \tau_2)$  where  $X$  is a set and  $\tau_1, \tau_2$  are topologies in  $X$ . A **quasi-pseudometric** in a set  $X$  is a function  $d : X \times X \rightarrow [0; +\infty)$  such that, for all  $x, y, z \in X$ ,  $d(x, y) \leq d(x, z) + d(z, y)$  and  $d(x, x) = 0$ . A quasi-pseudometric  $d$  in  $X$  is called a **quasi-metric** if, for all  $x, y \in X$ , the condition  $d(x, y) = 0$  implies  $x = y$  (cf. [Kel], [FL]).

Let  $d$  be a quasi-pseudometric in  $X$ . **The conjugate** of  $d$  is the quasi-pseudometric  $d^{-1}$  defined by  $d^{-1}(x, y) = d(y, x)$  for  $x, y \in X$ . The  $d$ -ball with centre  $x \in X$  and radius  $r \in (0; +\infty)$  is the set  $B_d(x, r) = \{y \in X : d(x, y) < r\}$ . The collection  $\tau(d) = \{V \subseteq X : \forall x \in V \exists n \in \omega B_d(x, \frac{1}{2^n}) \subseteq V\}$  is **the topology in  $X$  induced by  $d$** . The triple  $(X, \tau(d), \tau(d^{-1}))$  is **the bitopological space associated with  $d$** .

**Definition 1.1.** A bitopological space  $(X, \tau_1, \tau_2)$  is **(quasi)-metrizable** if there exists a (quasi)-metric  $d$  in  $X$  such that  $\tau_1 = \tau(d)$  and  $\tau_2 = \tau(d^{-1})$  (cf. pp. 74–75 of [Kel]).

One can find a considerable number of quasi-metrization theorems in [An] and in other sources (cf. [FL]).

We recall that, according to [Al]–[Hu], a **boundedness** in a set  $X$  is a (non-void) ideal of subsets of  $X$ . A boundedness  $\mathcal{B}$  in  $X$  is called a **bornology** in  $X$  if every singleton of  $X$  is a member of  $\mathcal{B}$  (cf. 1.1.1 in [H-N]).

**Definition 1.2** (cf. Definition 4.1 of [Hu]). If  $\mathcal{B}$  is a boundedness in  $X$ , then a collection  $\mathcal{A}$  is called a **base** for  $\mathcal{B}$  if  $\mathcal{A} \subseteq \mathcal{B}$  and every set of  $\mathcal{B}$  is a subset of a member of  $\mathcal{A}$ . A **second-countable boundedness** is a boundedness which has a countable base.

**Definition 1.3.** Let  $(X, \tau_1, \tau_2)$  be a bitopological space. A boundedness  $\mathcal{B}$  in  $X$  will be called  **$(\tau_1, \tau_2)$ -proper** if, for each  $A \in \mathcal{B}$ , there exists  $B \in \mathcal{B}$  such that  $\text{cl}_{\tau_2} A \subseteq \text{int}_{\tau_1}(B)$ . If  $\tau = \tau_1 = \tau_2$  and the boundedness  $\mathcal{B}$  is  $(\tau, \tau)$ -proper, we will say that  $\mathcal{B}$  is  **$\tau$ -proper**.

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Let us notice that if  $(X, \tau)$  is a topological space, then a boundedness  $\mathcal{B}$  in  $X$  is  $\tau$ -proper if and only if the universe  $((X, \tau), \mathcal{B})$  is proper in the sense of Definition 3.4 of [Hu].

**Definition 1.4.** (i) We say that a **bornological biuniverse** is an ordered pair  $((X, \tau_1, \tau_2), \mathcal{B})$  where  $(X, \tau_1, \tau_2)$  is a bitopological space and  $\mathcal{B}$  is a bornology in  $X$ .

(ii) A **bornological universe** is an ordered pair  $((X, \tau), \mathcal{B})$  where  $(X, \tau)$  is a topological space and  $\mathcal{B}$  is a bornology in  $X$  (cf. Definition 1.2 of [Hu]).

**Definition 1.5.** Let  $d$  be a quasi-metric in  $X$  and let  $A$  be a subset of  $X$ . Then:

- (i)  $A$  is called  **$d$ -bounded** if there exist  $x \in X$  and  $r \in (0; +\infty)$  such that  $A \subseteq B_d(x, r)$ ;
- (ii) if  $A$  is not  $d$ -bounded, we say that  $A$  is  **$d$ -unbounded**;
- (iii)  $\mathcal{B}_d(X)$  is the collection of all  $d$ -bounded subsets of  $X$ .

For a quasi-metric  $d$  in  $X$ , a set  $A \subseteq X$  can be simultaneously  $d$ -bounded and  $d^{-1}$ -unbounded.

**Example 1.6.** For  $x, y \in \omega$ , let  $d(x, y) = 0$  if  $x = y$  and  $d(x, y) = 2^x$  if  $x \neq y$ . Then  $\omega = B_d(0, 2)$ , so  $\omega$  is  $d$ -bounded. However, for arbitrary  $x \in \omega$  and  $r \in (0; +\infty)$ , if  $y \in \omega$  is such that  $2^y > r$ , then  $y \notin B_{d^{-1}}(x, r)$ . Therefore,  $\omega$  is  $d^{-1}$ -unbounded.

**Definition 1.7.** We say that a bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$  is **(quasi)-metrizable** if there exists a (quasi)-metric  $d$  in  $X$  such that  $\tau_1 = \tau(d)$ ,  $\tau_2 = \tau(d^{-1})$  and  $\mathcal{B} = \mathcal{B}_d(X)$ .

It is obvious that if  $\tau$  is a topology in  $X$ , then a bornological biuniverse  $((X, \tau, \tau), \mathcal{B})$  is metrizable if and only if the bornological universe  $((X, \tau), \mathcal{B})$  is metrizable in the sense of Definition 10.1 of [Hu]. Let us recall this definition.

**Definition 1.8.** Let  $((X, \tau), \mathcal{B})$  be a bornological universe. We say that:

- (i)  $((X, \tau), \mathcal{B})$  is **metrizable (in the sense of Hu)** if there exists a metric  $d$  on  $X$  such that  $\tau = \tau(d)$  and  $\mathcal{B} = \{A \subseteq X : \text{diam}_d(A) < +\infty\}$  where  $\text{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}$ ;
- (ii)  $((X, \tau), \mathcal{B})$  is **quasi-metrizable** if there exists a quasi-metric  $d$  on  $X$  such that  $\tau = \tau(d)$  and, moreover,  $\mathcal{B}$  is the collection of all  $d$ -bounded sets.

We show in Section 4 that if a bornological biuniverse  $((X, \tau, \tau), \mathcal{B})$  is quasi-metrizable, then the bornological universe  $((X, \tau), \mathcal{B})$  is metrizable.

Of course, it is impossible to prove anything in mathematics without axioms. The basic set-theoretic system of axioms used in this paper is **ZF** (cf. [Ku1]-[Ku2]). If a relatively independent of **ZF** axiom **A** is added to **ZF**, we shall write **ZF** + **A** and clearly denote our theorems proved in **ZF** + **A** but not in **ZF**. As far as set-theoretic axioms are concerned, we use standard notation from [Ku2] and [Her]. In particular, we denote **ZF** + **AC** by **ZFC**. If it is necessary, we use a modification of **ZF** signalled in [PW].

According to Theorem 1 of [Vr2] and Theorem 13.2 of [Hu], the following theorem can be called **Hu's metrization theorem for bornological universes**:

**Theorem 1.9.** *It holds true in **ZFC** that a bornological universe  $((X, \tau), \mathcal{B})$  is metrizable if and only if it is proper, while, simultaneously,  $(X, \tau)$  is metrizable and  $\mathcal{B}$  has a countable base.*

One of the main aims of our present work is to show that the proof to Hu's metrization theorem in [Hu] highly involves the axiom **CC** of countable choice and to prove in **ZF** the following generalization of Theorem 1.9:

**Theorem 1.10.** *It is true in **ZF** that a bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$  is quasi-metrizable if and only if  $\mathcal{B}$  has a countable base and it is  $(\tau_1, \tau_2)$ -proper, while the bitopological space  $(X, \tau_1, \tau_2)$  is quasi-metrizable.*

We deduce Theorem 1.9 from 1.10 and we prove a stronger theorem than 1.10 in Section 4 (Theorem 4.7). We also give some other applications of Theorem 1.10. Especially in Sections 2, 3 and 7, we give examples of unprovable in **ZF** results on bornological universes that were obtained by other authors probably either in **ZFC** or in naive preaxiomatic set theory. Section 5 contains a generalization of Theorem 13.5 of [Hu]. We pay a special attention to [GM] and, in Section 6, we modify the basic theorem of [GM] to get

necessary and sufficient conditions for a bornological quasi-metric universe to be uniformly quasi-metrizable (Theorem 6.5); furthermore, in Section 8, we modify a theorem about compact bornologies from [GM]. Finally, in Section 10, we offer relevant to bornologies concepts of quasi-metrizability for generalized topological spaces in the sense of Delfs and Knebusch (cf. [DK], [P1], [P2], [PW]) and give a number of illuminating examples. Section 9 concerns bornologies in generalized topological spaces and it is a preparation for Section 10. We close the paper with Section 11 where there are remarks about new topological categories.

We recommend [En] as a monograph on topology that we use. Our basic knowledge about category theory is taken from [AHS]. Models of set theory applied by us are described in [Her], [J1]-[J2] and [HR].

## 2 Countability

The axiom of countable choice is usually denoted by **CC**, **ACC** or **CAC**.

We shall use the following standard notions of finiteness and infinity:

**Definition 2.1.** A set  $X$  is called:

- (i) **finite** or **T-finite** (truly finite) if there exists  $n \in \omega$  such that  $X$  is equipollent with  $n$ ;
- (ii) **D-finite** or **Dedekind-finite** if no proper subset of  $X$  is equipollent with  $X$ .
- (iii) **infinite** or **T-infinite** if it is not finite, and **D-infinite** if it is not D-finite.

A set is T-finite if and only if it is finite in Tarski's sense (cf. Definition 4.4 of [Her]). Other notions relevant to finiteness were studied, for example, in [Cruz]. The term *truly finite* was suggested by K. Kunen in a private communication with E. Wajch.

Let us establish three distinct notions of countability.

**Definition 2.2.** A set  $X$  is called:

- (i) **countable** or **T-countable** (truly countable) if  $X$  is equipollent with a subset of  $\omega$ ;

- (ii) **D-countable** if every D-infinite subset of  $X$  is equipollent with  $X$ ;
- (iii) **W-countable** if every well-orderable subset of  $X$  is D-countable.

To each notion of countability  $Q$  corresponds a notion of uncountability.

**Definition 2.3.** We say that a set is **Q-uncountable** if it is not  $Q$ -countable where  $Q$  stands for T, D or W. Sets that are T-uncountable are called **uncountable**.

Let us denote by **CC**(D-fin) the following statement: every non-void countable collection of pairwise disjoint non-void D-finite sets has a choice function. As usual, **CC**(fin) is the statement: every non-void countable collection of pairwise disjoint non-void finite sets has a choice function.

**Proposition 2.4.** *The following conditions are equivalent:*

- (i) **CC**(D-fin);
- (ii) every D-countable set is countable.

*Proof.* Let  $X$  be a set. Assume that  $X$  is D-countable. If  $X$  is D-infinite, then  $X$  is countable (cf. [W], p. 48). Assume that  $X$  is D-finite. Then if (i) holds, it follows from E13 of Section 4.1 of [Her] that the set  $X$  is finite, so countable. Hence (i) implies (ii). Now, assume that (ii) holds and that  $X$  is D-finite. Then  $X$  is D-countable, so countable. This implies that  $X$  is equipollent with a finite subset of  $\omega$  and, in consequence,  $X$  is finite. By E13 of Section 4.1 of [Her], (ii) implies (i).  $\square$

**Fact 2.5.** *For every D-finite set  $X$ , the following conditions are equivalent:*

- (i)  $X$  is finite;
- (ii)  $X \cup \omega$  is D-countable.

**Corollary 2.6.** *If  $X$  is an infinite D-finite set, then the set  $X \cup \omega$  is D-uncountable.*

**Fact 2.7.** *A set  $X$  is countable if and only if  $X \cup \omega$  is D-countable.*

**Corollary 2.8.** *In every model  $\mathbf{M}$  for **ZF** such that there is in  $\mathbf{M}$  an infinite D-finite subset of  $\mathbb{R}$ , the collection of all D-countable subsets of  $\mathbb{R}$  is not a bornology.*

**Fact 2.9.** *In every set  $X$ , the following collections are bornologies:*

- (i) *the collection  $\mathbf{FB}(X)$  of all finite subsets of  $X$ ;*
- (ii) *the collection of all  $D$ -finite subsets of  $X$ ;*
- (iii) *the collection of all countable subsets of  $X$ ;*
- (iv) *the collection of all  $W$ -countable subsets of  $X$ .*

Several remarks on  $D$ -countability can be found in [W].

### 3 Second-countable bornological biuniverses

One may deduce wrongly from Theorem 5.5 of [Hu] that every base of a second-countable boundedness  $\mathcal{B}$  certainly contains a countable base for  $\mathcal{B}$ . However, we are going to prove that Theorem 5.5 of [Hu] is relatively independent of  $\mathbf{ZF}$ . To do this, let us consider the following bornologies in  $\mathbb{R}$ :

$$\begin{aligned}\mathbf{UB}(\mathbb{R}) &= \{A \subseteq \mathbb{R} : \exists_{r \in \mathbb{R}} A \subseteq (-\infty; r)\}, \\ \mathbf{LB}(\mathbb{R}) &= \{A \subseteq \mathbb{R} : \exists_{r \in \mathbb{R}} A \subseteq (r; +\infty)\}.\end{aligned}$$

Of course,  $\mathbf{UB}(\mathbb{R})$  and  $\mathbf{LB}(\mathbb{R})$  are second-countable.

**Proposition 3.1.** *Equivalent are:*

- (i)  $\mathbf{CC}(\mathbb{R})$ ;
- (ii) *for every unbounded to the right subset  $D$  of  $\mathbb{R}$ , the collection  $\mathcal{A}(D) = \{(-\infty; d) : d \in D\}$  contains a countable base for  $\mathbf{UB}(\mathbb{R})$ ;*
- (iii) *for every unbounded to the left subset  $D$  of  $\mathbb{R}$ , the collection  $\mathcal{A}(D) = \{(d; +\infty) : d \in D\}$  contains a countable base for  $\mathbf{LB}(\mathbb{R})$ ;*

*Proof.* First, assume that  $\mathbf{CC}(\mathbb{R})$  holds and that  $D$  is an unbounded to the right subset of  $\mathbb{R}$ . It follows from Theorem 3.8 of [Her] that there exists an unbounded sequence  $(d_n)_{n \in \omega}$  of elements of  $D \cap [0; +\infty)$ . Then  $\{(-\infty; d_n) : n \in \omega\}$  is a countable base for  $\mathbf{UB}(\mathbb{R})$ .

Now, suppose that  $\mathbf{CC}(\mathbb{R})$  does not hold. By Theorem 3.8 of [Her], there exists an unbounded subset  $B$  of  $\mathbb{R}$  which does not contain any unbounded sequence. Then the set  $D = B \cup \{-x : x \in B\}$  does not contain any

unbounded sequence. The collection  $\mathcal{A}(D)$  is a base for  $\mathbf{UB}(\mathbb{R})$  such that  $\mathcal{A}(D)$  does not contain any countable base for  $\mathbf{UB}(\mathbb{R})$ . Hence (i) implies (ii).

To show that (ii) and (iii) are equivalent, it suffices to make a suitable use of the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = -x$  for  $x \in \mathbb{R}$ .  $\square$

**Corollary 3.2.** *Let  $\mathbf{M}$  be any model for  $\mathbf{ZF}$  such that  $\mathbf{CC}(\mathbb{R})$  fails in  $\mathbf{M}$ . Then the bornology  $\mathbf{UB}(\mathbb{R})$  has a base which does not contain any countable base for  $\mathbf{UB}(\mathbb{R})$ . In consequence, Theorem 5.5 of [Hu] is false in  $\mathbf{M}$ .*

**Proposition 3.3.** *( $\mathbf{ZF} + \mathbf{CC}$ ) If a boundedness  $\mathcal{B}$  in  $X$  has a countable base, then every base for  $\mathcal{B}$  contains a countable base for  $\mathcal{B}$ .*

*Proof.* Let  $\mathcal{A}$  be a base for  $\mathcal{B}$ . Consider an arbitrary countable base  $\mathcal{C}$  for  $\mathcal{B}$ . Then  $\mathcal{C} \neq \emptyset$ . For  $C \in \mathcal{C}$ , let  $\mathcal{A}(C) = \{A \in \mathcal{A} : C \subseteq A\}$ . Since  $\mathcal{A}$  is a base for  $\mathcal{B}$ , we have  $\mathcal{A}(C) \neq \emptyset$  whenever  $C \in \mathcal{C}$ . Using  $\mathbf{CC}$ , we deduce that there exists  $x \in \prod_{C \in \mathcal{C}} \mathcal{A}(C)$ . Then  $\mathcal{A}_0 = \{x(C) : C \in \mathcal{C}\} \subseteq \mathcal{A}$  and  $\mathcal{A}_0$  is a countable base for  $\mathcal{B}$ .  $\square$

We can get the following correct modification in  $\mathbf{ZF}$  of Theorem 5.5 of [Hu]:

**Proposition 3.4.** *Let  $\mathcal{C}$  be a countable base for a boundedness  $\mathcal{B}$  in  $X$  such that  $\mathcal{B}$  does not have a maximal set with respect to inclusion. Then there exists a strictly increasing sequence  $(A_n)$  of members of  $\mathcal{C}$  such that the collection  $\{A_n : n \in \omega\}$  is a base for  $\mathcal{B}$ .*

*Proof.* It follows from the countability of  $\mathcal{C}$  that we can write  $\mathcal{C} = \{C_n : n \in \omega\}$ . Let  $A_0 = C_0$ . Since  $\mathcal{B}$  does not contain maximal bounded sets, there exists  $B \in \mathcal{B}$  such that  $B$  is not a subset of  $A_0 \cup C_1$  and there exists  $C \in \mathcal{C}$  such that  $A_0 \cup B \cup C_1 \subseteq C$ . This proves that there exists  $C \in \mathcal{C}$  such that  $A_0 \cup C_1 \neq C$  and  $A_0 \cup C_1 \subseteq C$ . Let  $n_1 = \min\{n \in \omega : A_0 \cup C_1 \subset C_n\}$  and  $A_1 = C_{n_1}$ . Of course, we use the symbol  $\subset$  for strict inclusion. Suppose that, for  $m \in \omega \setminus \{0\}$ , we have already defined the set  $A_m \in \mathcal{C}$ . In much the same way as above, we take  $n_{m+1} = \min\{n \in \omega : A_m \cup C_{m+1} \subset C_n\}$  and  $A_{m+1} = C_{n_{m+1}}$ . The sequence  $(A_n)$  has the required properties.  $\square$

Although Theorem 5.5 of [Hu] is unprovable in  $\mathbf{ZF}$ , the following proposition about bornological biuniverses clearly shows that Theorem 5.6 of [Hu] holds true in  $\mathbf{ZF}$ ; however, in the proof of Theorem 5.6 in [Hu], an illegal in  $\mathbf{ZF}$  countable choice was involved. Therefore, we offer its more careful proof in  $\mathbf{ZF}$ .



**Proposition 3.5.** *Let us suppose that  $(X, \tau_1, \tau_2)$  is a bitopological space, while  $\mathcal{B}$  is a second-countable  $(\tau_1, \tau_2)$ -proper boundedness in  $X$  such that  $\mathcal{B}$  does not have maximal sets with respect to inclusion. Then there exists a strictly increasing sequence  $(A_n)$  of  $\tau_1$ -open sets such that  $\mathcal{A} = \{A_n : n \in \omega\}$  is a base for  $\mathcal{B}$  such that  $cl_{\tau_2} A_n \subset A_{n+1}$  for each  $n \in \omega$ .*

*Proof.* Take, by Proposition 3.4, a strictly increasing countable base  $\mathcal{C} = \{C_n : n \in \omega\}$  for  $\mathcal{B}$ . Let  $A_0 = \text{int}_{\tau_1} C_0$ . Suppose that, for  $m \in \omega$ , we have already defined a  $\tau_1$ -open set  $A_m \in \mathcal{B}$ . We use similar arguments to the ones given in the proof to Proposition 3.4 with the exception that, since  $\mathcal{B}$  is  $(\tau_1, \tau_2)$ -proper, we may define  $n_{m+1} = \min\{n \in \omega : cl_{\tau_2}(A_m \cup C_{m+1}) \subset \text{int}_{\tau_1} C_n\}$  and  $A_{m+1} = \text{int}_{\tau_1} C_{n_{m+1}}$ .  $\square$

## 4 Quasi-metrization theorems for bornological biuniverses

If  $\tau$  is a topology on  $X$  and if  $A \subseteq X$ , we denote  $\tau|_A = \{A \cap V : V \in \tau\}$ . For the real line  $\mathbb{R}$ , the topology  $u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty; a) : a \in \mathbb{R}\}$  is called **the upper topology** on  $\mathbb{R}$ , while the topology  $l = \{\emptyset, \mathbb{R}\} \cup \{(a; +\infty) : a \in \mathbb{R}\}$  is called **the lower topology** on  $\mathbb{R}$  (cf. [FL], [Sal]). If  $A \subseteq \mathbb{R}$ , then we use  $(A, u, l)$  as an abbreviation of  $(A, u|_A, l|_A)$  where  $u = u|_A$  and  $l = l|_A$ .

**Definition 4.1.** Suppose that  $(X, \tau_1^X, \tau_2^X)$  and  $(Y, \tau_1^Y, \tau_2^Y)$  are bitopological spaces. A mapping  $f : X \rightarrow Y$  is called **bicontinuous with respect to**  $(\tau_1^X, \tau_2^X, \tau_1^Y, \tau_2^Y)$  (in abbreviation: bicontinuous) if

$$\{f^{-1}(V) : V \in \tau_i^Y\} \subseteq \tau_i^X$$

for each  $i \in \{1, 2\}$ .

A crucial role in the study of bornologies is played by a concept of a characteristic function of a bornology which is also called a forcing function (cf. [Hu], [Be]). We need to extend this concept to bornological biuniverses.

**Definition 4.2.** Let  $(X, \tau_1, \tau_2)$  be a bitopological space. Then a  $(\tau_1, \tau_2)$ -**characteristic function** for a bornology  $\mathcal{B}$  in  $X$ , is a bicontinuous function  $f : (X, \tau_1, \tau_2) \rightarrow ([0; +\infty), u, l)$  such that

$$\mathcal{B} = \{A \subseteq X : \sup\{f(x) : x \in A\} < +\infty\}.$$

**Fact 4.3** (cf. 4.1 of [Kel]). *Let  $d$  be a quasi-metric on  $X$  and let  $x_0 \in X$ . Define  $f(x) = d(x_0, x)$  for  $x \in X$ . Then the function  $f : (X, \tau(d), \tau(d^{-1})) \rightarrow ([0; +\infty), u, l)$  is bicontinuous.*

**Definition 4.4.** We say that a (quasi)-metric  $d$  **induces a bornological biuniverse**  $((X, \tau_1, \tau_2), \mathcal{B})$  if  $\tau_1 = \tau(d)$ ,  $\tau_2 = \tau(d^{-1})$  and  $\mathcal{B} = \mathcal{B}_d(X)$ .

**Proposition 4.5.** *Suppose that a bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$  is (quasi)-metrizable. Then there exists a  $(\tau_1, \tau_2)$ -characteristic function for the bornology  $\mathcal{B}$ .*

*Proof.* Let us consider an arbitrary point  $x_0 \in X$  and any (quasi)-metric  $d$  such that  $d$  induces the bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$ . Then, by Fact 4.3, a  $(\tau_1, \tau_2)$ -characteristic function for  $\mathcal{B}$  is the function  $f : X \rightarrow \mathbb{R}$  where  $f(x) = d(x_0, x)$  for  $x \in X$ .  $\square$

**Proposition 4.6.** *Suppose that a bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$  is such that  $\mathcal{B}$  has a  $(\tau_1, \tau_2)$ -characteristic function. Then  $\mathcal{B}$  is both second-countable and  $(\tau_1, \tau_2)$ -proper.*

*Proof.* Let  $f$  be any  $(\tau_1, \tau_2)$ -characteristic function for  $\mathcal{B}$ . For  $n \in \omega$ , let  $A_n = f^{-1}((-\infty, n])$ . Then the collection  $\{A_n : n \in \omega\}$  is a countable base for  $\mathcal{B}$  such that  $\text{cl}_{\tau_2} A_n \subseteq \text{int}_{\tau_1} A_{n+1}$ .  $\square$

**Theorem 4.7.** *Let us suppose that  $(X, \tau_1, \tau_2)$  is a (quasi)-metrizable bitopological space and that  $\mathcal{B}$  is a bornology in  $X$ . Then the following conditions are all equivalent:*

- (i) *the bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$  is (quasi)-metrizable;*
- (ii) *there exists a  $(\tau_1, \tau_2)$ -characteristic function for  $\mathcal{B}$ ;*
- (iii) *the bornology  $\mathcal{B}$  is  $(\tau_1, \tau_2)$ -proper and it has a countable base.*

*Proof.* Let us consider any quasi-metric  $\sigma$  on  $X$  such that  $\tau_1 = \tau(\sigma)$  and  $\tau_2 = \tau(\sigma^{-1})$ . Put  $d(x, y) = \min\{\sigma(x, y), 1\}$  for  $x, y \in X$ . It is easy to observe that if  $X \in \mathcal{B}$ , then all conditions (i) – (iii) are fulfilled. Assume that  $X \notin \mathcal{B}$ . It follows from Proposition 4.5 that (i) implies (ii). Assume (ii) and suppose that  $f$  is a  $(\tau_1, \tau_2)$ -characteristic function for  $\mathcal{B}$ . For  $x, y \in X$ , let  $\rho(x, y) = d(x, y) + \max\{f(y) - f(x), 0\}$ . Then the quasi-metric  $\rho$  induces the bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$ . In the case when  $\sigma$  is a metric,

we can put  $\rho(x, y) = d(x, y) + |f(y) - f(x)|$  to obtain a metric that induces  $((X, \tau_1, \tau_2), \mathcal{B})$ . Hence (ii) implies (i).

Now, assume that (iii) holds. Since  $X \notin \mathcal{B}$ , it follows from Proposition 3.5 that there exists a base  $\{A_n : n \in \omega\}$  for  $\mathcal{B}$  such that  $\text{cl}_{\tau_2} A_n$  is a proper subset of  $\text{int}_{\tau_1} A_{n+1}$  for each  $n \in \omega$ . We may assume that  $A_0 = \emptyset$ . For  $n \in \omega \setminus \{0\}$  and  $x \in X$ , let  $f_n(x) = d(\text{cl}_{\tau_2} A_n, x)$  and  $g_n(x) = d(x, X \setminus \text{int}_{\tau_1} A_{n+1})$ . Then  $f_n : (X, \tau_1, \tau_2) \rightarrow ([0; +\infty), u, l)$  and  $g_n : (X, \tau_1, \tau_2) \rightarrow ([0; +\infty), l, u)$  are bicontinuous. For each  $x \in X$ , we have  $f_n(x) + g_n(x) \neq 0$ , so we can put  $h_n(x) = \frac{f_n(x)}{f_n(x) + g_n(x)}$ . Moreover, we define  $h_0(x) = 1$  for each  $x \in X$ . It is easy to check that the function  $h_n : (X, \tau_1, \tau_2) \rightarrow ([0; 1], u, l)$  is bicontinuous for each  $n \in \omega$  (cf. the proof to Corollary 2.2.16 in [Sal]). Let  $\psi(x) = h_n(x) + n$  when  $x \in \text{int}_{\tau_1} A_{n+1} \setminus \text{int}_{\tau_1} A_n$ . We are going to prove that the function  $\psi$  is bicontinuous with respect to  $(\tau_1, \tau_2, u, l)$ .

Let  $x \in \text{int}_{\tau_1} A_{n+1} \setminus \text{int}_{\tau_1} A_n$  and  $y \in \text{int}_{\tau_1} A_{m+1} \setminus \text{int}_{\tau_1} A_m$ . Consider any real numbers  $r, s$  such that  $r < \psi(x) < s$ . We assume that  $n \neq 0$ . There exists  $U_s \in \tau_1$  such that  $x \in U_s \subseteq \text{int}_{\tau_1} A_{n+1}$  and if  $y \in U_s$ , then  $h_n(y) + n < s$ . There exists  $V_r \in \tau_2$  such that  $x \in V_r \subseteq X \setminus \text{cl}_{\tau_2} A_{n-1}$  and if  $y \in V_r$ , then  $h_n(y) + n > r$ . Of course, if  $m = n$ , then  $\psi(y) < s$  when  $y \in U_s$ , while  $\psi(y) > r$  when  $y \in V_r$ . Let us assume that  $m \neq n$ . Suppose that  $y \in U_s$ . Then  $m < n$ , so  $\psi(y) \leq 1 + m \leq n \leq \psi(x) < s$ .

Suppose that  $y \in V_r$ . If  $m > n$ , we have  $\psi(y) \geq m \geq 1 + n \geq \psi(x) > r$ . Let  $m < n$ . Since  $y \notin \text{int}_{\tau_1} A_{n-1}$ , we have  $m + 1 \geq n$ . As  $m + 1 \leq n$ , we have  $m + 1 = n$ . If  $x \notin \text{cl}_{\tau_2} A_n$  we could take  $V_r^* = V_r \cap (X \setminus \text{cl}_{\tau_2} A_n) \in \tau_2$  and observe that if  $y \in V_r^*$ , then  $m \geq n$  and  $\psi(y) > r$ . Let us consider the case when  $m < n$  and  $x \in \text{cl}_{\tau_2} A_n$ . Then  $\psi(x) = n$ . We take a positive real number  $\epsilon$  such that  $n - \epsilon > r$ . Since  $h_{n-1}(x) = 1$ , there exists  $W_\epsilon \in \tau_2$  such that  $x \in W_\epsilon$  and  $h_{n-1}(t) > 1 - \epsilon$  for each  $t \in W_\epsilon$ . If  $y \in W_\epsilon \cap V_r$  and  $m + 1 = n$ , then  $\psi(y) = h_{n-1}(y) + n - 1 > 1 - \epsilon + n - 1 > r$ . The case when  $n = 0$  is also obvious now. This completes the proof that  $\psi$  is bicontinuous with respect to  $(\tau_1, \tau_2, u, l)$ . It is easy to check that  $\mathcal{B} = \{A \subseteq X : \sup \psi(A) < +\infty\}$ , so  $\psi$  is a  $(\tau_1, \tau_2)$ -characteristic function for  $\mathcal{B}$ . Hence (ii) follows from (iii). To complete the proof, it suffices to apply Proposition 4.6.  $\square$

**Corollary 4.8.** *Theorem 1.10 is true.*

**Corollary 4.9.** *The assumption of ZFC can be weakened to ZF in Theorem 1.9.*

**Corollary 4.10.** *Let us suppose that  $(X, \tau)$  is a topological space and  $\mathcal{B}$  is*

a bornology in  $X$ . Then the bornological biuniverse  $((X, \tau, \tau), \mathcal{B})$  is quasi-metrizable if and only if the bornological universe  $((X, \tau), \mathcal{B})$  is metrizable.

*Proof.* It suffices to prove that if there exists a quasi-metric which induces  $((X, \tau, \tau), \mathcal{B})$ , then  $((X, \tau), \mathcal{B})$  is metrizable. Let  $d$  be a quasi-metric in  $X$  such that  $\tau = \tau(d) = \tau(d^{-1})$  and  $\mathcal{B} = \mathcal{B}_d(X)$ . Define  $\rho = \max\{d, d^{-1}\}$ . Then  $\rho$  is a metric in  $X$  such that  $\tau(\rho) = \tau$ . Moreover, by Theorem 4.7, the bornology  $\mathcal{B}$  is second-countable and  $\tau$ -proper; hence, the bornological universe  $((X, \tau), \mathcal{B})$  is metrizable by Theorem 4.7.  $\square$

**Example 4.11.** Let  $d$  be the quasi-metric from Example 1.6. Then  $\tau(d) = \tau(d^{-1}) = \mathcal{P}(\omega)$ . Moreover,  $\mathcal{B}_d(\omega) = \mathcal{P}(\omega)$  and  $\mathcal{B}_{d^{-1}}(\omega) = \mathbf{FB}(\omega)$ . The metric  $\rho = \max\{d, d^{-1}\}$  does not induce  $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathcal{B}_d(\omega))$ ; however,  $\rho$  induces  $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathbf{FB}(\omega))$ .

**Example 4.12.** Let  $\tau_{S,r}$  be the right half-open interval topology in  $\mathbb{R}$  and let  $\tau_{S,l}$  be the left half-open interval topology in  $\mathbb{R}$ . Then  $(\mathbb{R}, \tau_{S,r})$  is the Sorgenfrey line.

- (i) The bornological biuniverse  $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{UB}(\mathbb{R}))$  is not metrizable but it is quasi-metrizable by the following quasi-metric  $\rho_S$ :

$$\rho_S(x, y) = \begin{cases} y - x, & x \leq y \\ 1, & x > y. \end{cases}$$

Let us notice that  $\mathcal{B}_{\rho_S^{-1}}(\mathbb{R}) = \mathbf{LB}(\mathbb{R})$  and the quasi-metric  $\rho_S$  does not induce the bornological biuniverse  $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$ . However, the bornological biuniverse  $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$  is induced by the quasi-metric  $\rho_L$  defined as follows:

$$\rho_L(x, y) = \begin{cases} \min\{y - x, 1\}, & x \leq y \\ 1 + x - y, & x > y. \end{cases}$$

- (ii) The non-metrizable bornological biuniverse  $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{P}(\mathbb{R}))$  is quasi-metrizable by the quasi-metric  $\rho_{S,1}$  defined as follows:

$$\rho_{S,1}(x, y) = \begin{cases} \min\{1, y - x\}, & x \leq y \\ 1, & x > y. \end{cases}$$

**Example 4.13.** We consider the following **hedgehog-like scheme**. Let  $(X, d)$  be a quasi-metric space such that  $X$  has at least two distinct points. Let  $S$  be a non-empty set. We fix  $x_0 \in X$  and put  $Y_s = (X \setminus \{x_0\}) \times \{s\}$  for  $s \in S$ . Let us fix  $p \notin \bigcup_{s \in S} Y_s$  and put  $Y = \{p\} \cup \bigcup_{s \in S} Y_s$ . Let  $x, y \in X \setminus \{x_0\}$  and  $s, s' \in S$ . We define  $\rho(p, p) = 0$ ,  $\rho((x, s), p) = d(x, x_0)$ ,  $\rho(p, (x, s)) = d(x_0, x)$  and  $\rho((x, s), (y, s)) = d(x, y)$ . If  $s \neq s'$ , we put  $\rho((x, s), (y, s')) = d(x, x_0) + d(x_0, y)$ . Let us consider the collection  $\mathcal{B}$  of all sets  $A \subseteq Y$  such that there are finite  $S(A) \subseteq S$  such that  $A \subseteq \{p\} \cup \bigcup_{s \in S(A)} Y_s$ . Then  $\mathcal{B}$  is a bornology in  $Y$ . If  $S$  is countable, then  $\mathcal{B}$  is second-countable. If  $S$  is infinite and, simultaneously,  $x_0$  is an accumulation point of  $(X, \tau(d))$ , then the bornology  $\mathcal{B}$  is not  $(\tau(\rho), \tau(\rho^{-1}))$ -proper. Let us denote the bornological biuniverse  $((Y, \tau(\rho), \tau(\rho^{-1})), \mathcal{B})$  by  $J(X, d, x_0, S)$  and let  $Y(X, d, x_0, S) = (Y, \tau(\rho))$ . We can apply  $J(X, d, x_0, S)$  as follows.

- (i) If  $X = [0; 1]$  and  $d(x, y) = |x - y|$  for  $x, y \in X$ , then the bornological universe  $J(X, d, 0, \omega)$  is not quasi-metrizable although its bornology is second-countable. In this case,  $Y(X, d, x_0, \omega)$  is the hedgehog space of spininess  $\omega$  (cf. 4.1.5 of [En]), so we can call  $((Y, \tau(\rho)), \mathcal{B})$  **the bornological hedgehog space of spininess  $\omega$** .
- (ii) If  $\rho_S$  is the quasi-metric defined in Example 4.12 (i), then the bornological biuniverse  $J(\mathbb{R}, \rho_S, 0, \omega)$  is not quasi-metrizable but its bornology has a countable base.
- (iii) Let  $C$  be the unit circle in  $\mathbb{R}^2$ . We fix  $x_0 \in C$  and we consider the Euclidean metric  $d_e$  in  $C$ . The bornological biuniverse  $J(C, d_e, x_0, \omega)$  is not quasi-metrizable although its bornology has a countable base. We can call  $J(C, d_e, x_0, \omega)$  **the bornological metric wedge sum of circles**. In this case, the topological space  $Y(C, d_e, x_0, \omega)$  is not compact.
- (iv) It is worthwhile to compare  $J(C, d_e, x_0, \omega)$  with **the bornological Hawaiian earring**  $(H, \mathcal{B}_H)$  where  $H = \bigcup_{n \in \omega \setminus \{0\}} H_n$  is considered with its natural topology inherited from  $\mathbb{R}^2$  and, for each  $n \in \omega \setminus \{0\}$ , the set  $H_n$  is the circle with centre  $(\frac{1}{n}, 0)$  and radius  $\frac{1}{n}$ , while  $\mathcal{B}_H$  is the collection of all sets  $A \subseteq H$  such that there exist sets  $n(A) \in \omega$  such that  $A \subseteq \bigcup_{n \in n(A) \setminus \{0\}} H_n$ . Then  $H$  is compact and the bornology  $\mathcal{B}_H$  has a countable base. Since there does not exist  $A \in \mathcal{B}_H$  such that

$(0, 0) \in \text{int}_{d_e} A$ , it follows from Theorem 4.7 that the bornological universe  $(H, \mathcal{B}_H)$  is not quasi-metrizable.

In view of the examples above, when  $d$  is a quasi-metric in  $X$  and  $\mathcal{B}$  is a bornology in  $X$  but  $d$  does not induce the bornological biuniverse  $(X, \mathcal{B}) = ((X, \tau(d), \tau(d^{-1})), \mathcal{B})$ , it might be interesting to find, in terms of  $d$ , necessary and sufficient conditions for  $(X, \mathcal{B})$  to be quasi-metrizable. To do this, we need the following concept:

**Definition 4.14.** Let  $d$  be a quasi-pseudometric in a set  $X$  and let  $\delta \in (0; +\infty)$ . For a set  $A \subseteq X$ , the  $\delta$ -neighbourhood of  $A$  with respect to  $d$  is the set  $[A]_d^\delta = \bigcup_{a \in A} B_d(a, \delta)$ .

Let us notice that if  $\emptyset \neq A \subseteq X$ , then  $[A]_d^\delta = \{x \in X : d(A, x) < \delta\}$ .

**Theorem 4.15.** For every bornological biuniverse  $((X, \tau_1, \tau_2), \mathcal{B})$ , the following conditions are equivalent:

- (i)  $((X, \tau_1, \tau_2), \mathcal{B})$  is (quasi)-metrizable;
- (ii) there exists a (quasi)-metric  $d$  in  $X$  such that  $\tau_1 = \tau(d)$ ,  $\tau_2 = \tau(d^{-1})$  and  $\mathcal{B}$  has a base  $\{B_n : n \in \omega\}$  with the following property:

$$\forall n \in \omega \exists \delta \in (0; +\infty) [B_n]_d^\delta \subseteq B_{n+1}.$$

*Proof.* Let  $((X, \tau_1, \tau_2), \mathcal{B})$  be a bornological biuniverse. Suppose that (i) holds and that  $d$  is a (quasi)-metric in  $X$  such that  $d$  induces  $((X, \tau_1, \tau_2), \mathcal{B})$ . We consider an arbitrary  $x_0 \in X$  and, for  $n \in \omega$ , we define  $B_n = B_d(x_0, n+1)$ . Since  $[B_n]_d^{\frac{1}{2}} \subseteq B_{n+1}$ , we infer that (ii) follows from (i).

Assume that (ii) is satisfied. Let  $C \subseteq X$  and  $D \subseteq X$  be such that, for some  $\delta \in (0; +\infty)$ , the inclusion  $[C]_d^\delta \subseteq D$  holds. Let  $x \in \text{cl}_{\tau_2} C$ . There exists  $y \in C \cap B_{d^{-1}}(x, \delta)$ . Then  $d(y, x) < \delta$ , so  $x \in [C]_d^\delta$ . Therefore  $\text{cl}_{\tau_2} C \subseteq [C]_d^\delta$ . Of course, since  $[C]_d^\delta \subseteq D$ , we have  $[C]_d^\delta \subseteq \text{int}_{\tau_1} D$ . In consequence,  $\text{cl}_{\tau_2} C \subseteq \text{int}_{\tau_1} D$ . Now, we deduce from Theorem 4.7 that (ii) implies (i).  $\square$

**Corollary 4.16.** For every bornological universe  $((X, \tau), \mathcal{B})$ , the following conditions are equivalent:

- (i) the universe  $((X, \tau), \mathcal{B})$  is (quasi)-metrizable;

(ii) there exists a (quasi)-metric  $d$  in  $X$  such that  $\tau = \tau(d)$  and, simultaneously,  $\mathcal{B}$  has a base  $\{B_n : n \in \omega\}$  with the following property:

$$\forall_{n \in \omega} \exists_{\delta \in (0; +\infty)} [B_n]_d^\delta \subseteq B_{n+1};$$

The following example shows that the sets  $B_n$  can be  $d$ -unbounded in Theorem 4.15 and Corollary 4.16.

**Example 4.17.** For the bornological biuniverse  $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$  and for the quasi-metric  $\rho_S$  from Example 4.12, the sets  $B_n = [-n; +\infty)$  with  $n \in \omega$  satisfy condition (ii) of Theorem 4.15 if we put  $d = \rho_S$  and  $\delta = 1$ . However, the sets  $B_n$  are all  $\rho_S$ -unbounded.

We offer a number of other relevant examples in Section 10.

## 5 The kernel of a boundedness

If  $X$  is a topological space and  $\mathcal{B}$  is a boundedness in  $X$ , a notion of a kernel of the universe  $(X, \mathcal{B})$  was introduced in Definition 6.3 in [Hu]. We adapt this notion to our needs.

**Definition 5.1.** Let  $\tau$  be a topology in a set  $X$ . If  $\mathcal{B}$  is a boundedness in  $X$ , then the  $\tau$ -**kernel** of  $\mathcal{B}$  is the set

$$\Lambda_\tau(\mathcal{B}) = \bigcup \{\text{int}_\tau A : A \in \mathcal{B}\}.$$

**Definition 5.2.** Let  $(X, \tau_1, \tau_2)$  be a bitopological space. Suppose that  $\mathcal{B}$  is a boundedness in  $X$  and put  $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$ . Let  $\mathcal{B}_\Lambda = \{A \cap \Lambda : A \in \mathcal{B}\}$ . Then the ordered pair  $((\Lambda, \tau_1|_\Lambda, \tau_2|_\Lambda), \mathcal{B}_\Lambda)$  will be called **the bornological biuniverse induced by  $\mathcal{B}$** .

**Fact 5.3.** Suppose that  $(X, \tau_1, \tau_2)$  is a bitopological space. If  $\mathcal{B}$  is a  $(\tau_1, \tau_2)$ -proper boundedness in  $X$  and if  $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$ , then the bornology  $\mathcal{B}_\Lambda$  in  $\Lambda$  is  $(\tau_1|_\Lambda, \tau_2|_\Lambda)$ -proper.

For a topological space  $X = (X, \tau)$  and a boundedness  $\mathcal{B}$  in  $X$ , when  $\Lambda = \Lambda_\tau(\mathcal{B})$ , Theorem 13.5 of [Hu] concerns the problem of the metrizability of the bornological universe  $(\Lambda, \mathcal{B}_\Lambda)$  under the assumption that  $\Lambda$  is a separable metrizable subspace of  $X$ . However, the proof to Theorem 13.5 in [Hu] is not in **ZF**. We give a generalization to bornological universes of Theorem 13.5 of [Hu] and show its proof in **ZF**. We also show that the assumption of separability is needless in Theorem 13.5 of [Hu].

**Theorem 5.4.** *Assume that  $(X, \tau_1, \tau_2)$  is a bitopological space and that  $\mathcal{B}$  is a second-countable  $(\tau_1, \tau_2)$ -proper boundedness in  $X$ . Let  $\Lambda$  be the  $\tau_1$ -kernel of  $\mathcal{B}$  and suppose that the bitopological space  $(\Lambda, \tau_1|_\Lambda, \tau_2|_\Lambda)$  is quasi-metrizable. Then there exists a quasi-metric  $\rho$  on  $\Lambda$  such that the following conditions are satisfied:*

- (i)  $\tau_1|_\Lambda = \tau(\rho)$  and  $\tau_2|_\Lambda = \tau(\rho^{-1})$ ;
- (ii)  $\mathcal{B}$  is the collection of all  $\rho$ -bounded subsets of  $\Lambda$ ;
- (iii) for each pair of points  $x_0 \in \Lambda$ ,  $x_* \in X \setminus \Lambda$  and for each positive real number  $b$ , there exists  $G \in \tau_2$  such that  $x_* \in G$  and  $\rho(x_0, x) > b$  whenever  $x \in G \cap \Lambda$ .

*Proof.* Since  $\mathcal{B}$  is  $(\tau_1, \tau_2)$ -proper, we have  $\mathcal{B} = \mathcal{B}_\Lambda$ . In view of Fact 5.3,  $\mathcal{B}$  is  $(\tau_1|_\Lambda, \tau_2|_\Lambda)$ -proper. In the light of Theorem 4.7, there exists in **ZF** a quasi-metric  $\rho$  in  $\Lambda$  such that both conditions (i) and (ii) are satisfied. Let  $x_0 \in \Lambda$  and  $x_* \in X \setminus \Lambda$ . Consider an arbitrary positive real number  $b$ . Put  $B = \{x \in \Lambda : \rho(x_0, x) \leq b\}$ . Of course,  $B \in \mathcal{B}$ . Since  $\mathcal{B}$  is  $(\tau_1, \tau_2)$ -proper, there exists  $U \in \tau_1$  such that  $B \subseteq U$ . Using the assumption that  $\mathcal{B}$  is  $(\tau_1, \tau_2)$ -proper once again, we deduce that  $\text{cl}_{\tau_2} U \subseteq \Lambda$ . Let  $G = X \setminus \text{cl}_{\tau_2} U$ . Then  $G \in \tau_2$ ,  $G \cap \Lambda \subseteq \Lambda \setminus B$  and  $x_* \in G$ . It is evident that if  $x \in G \cap \Lambda$ , then  $\rho(x_0, x) > b$ .  $\square$

Now, we can immediately deduce in **ZF** the following improvement of Theorem 13.5 of [Hu]:

**Corollary 5.5.** *If  $\mathcal{B}$  is a second-countable proper boundedness in a topological space  $X$  such that the set  $\Lambda = \bigcup \mathcal{B}$  is a metrizable subspace of  $X$ , then there exists a metric  $\rho$  on  $\Lambda$  such that the following conditions are satisfied:*

- (i) the topology of  $\Lambda$  as a subspace of  $X$  is induced by  $\rho$ ;
- (ii)  $\mathcal{B} = \{A \subseteq \Lambda : \text{diam}_\rho(A) < +\infty\}$ ;
- (iii) for each pair of points  $x_0 \in \Lambda$ ,  $x_* \in X \setminus \Lambda$  and for each positive real number  $b$ , there exists an open set  $G$  in  $X$  such that  $x_* \in G$  and  $\rho(x_0, x) > b$  whenever  $x \in G \cap \Lambda$ .



## 6 Uniformly quasi-metrizable bornologies

This section has been inspired by the necessary and sufficient conditions for a bornology to be uniformly metrizable given in [GM]. We adapt the conditions of Theorem 2.4 of [GM] to bornologies in quasi-metric spaces.

For  $x, y \in \mathbb{R}$ , let us put

$$\rho_u(x, y) = \max\{y - x, 0\}, \rho_l(x, y) = \max\{x - y, 0\}.$$

Then  $\rho_u, \rho_l$  are quasi-pseudometrics in  $\mathbb{R}$  such that  $\rho_u^{-1} = \rho_l$ ; moreover,  $\tau(\rho_u)$  is the upper topology  $u$  in  $\mathbb{R}$ , while  $\tau(\rho_l)$  is the lower topology  $l$  in  $\mathbb{R}$ .

**Definition 6.1.** Let  $d_X, d_Y$  be quasi-pseudometrics in sets  $X$  and  $Y$ , respectively. We say that a mapping  $f : X \rightarrow Y$  is  $(d_X, d_Y)$ -**uniformly continuous** if the following condition is satisfied:

$$\forall \epsilon \in (0; +\infty) \exists \delta \in (0; +\infty) \forall x_1, x_2 \in X [d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon].$$

**Definition 6.2.** Quasi-pseudometrics  $d_0, d_1$  in a set  $X$  are called **uniformly equivalent** if the following condition holds:

$$\forall \epsilon \in (0; +\infty) \exists \delta_0, \delta_1 \in (0; +\infty) \forall x, y \in X \forall i \in \{0, 1\} [d_i(x, y) < \delta_i \Rightarrow d_{1-i}(x, y) < \epsilon].$$

**Definition 6.3.** Suppose that  $(X, d)$  is a quasi-metric space and that  $\mathcal{B}$  is a bornology in  $X$ . We say that  $\mathcal{B}$  is **uniformly quasi-metrizable with respect to  $d$**  if there exists a quasi-metric  $\rho$  in  $X$  such that  $d$  and  $\rho$  are uniformly equivalent, while  $\mathcal{B}$  is the collection of all  $\rho$ -bounded sets.

**Definition 6.4.** We say that quasi-metrics  $d, \rho$  in  $X$  are **uniformly locally identical** if they are uniformly equivalent and there exists  $\delta \in (0; +\infty)$  such that, for all  $x, y \in X$ , we have  $\rho(x, y) = d(x, y)$  whenever  $d(x, y) < \delta$  (cf. [WJ] and Remark 2.5 of [GM]).

**Theorem 6.5.** *Suppose that  $(X, d)$  is a quasi-metric space and that  $\mathcal{B}$  is a bornology in  $X$ . Then the following conditions are all equivalent:*

- (i)  $\mathcal{B}$  is uniformly quasi-metrizable with respect to  $d$ ;
- (ii)  $\mathcal{B}$  has a base  $\{B_n : n \in \omega\}$  such that, for some  $\delta \in (0; +\infty)$  and for each  $n \in \omega$ , the inclusion  $[B_n]_d^\delta \subseteq B_{n+1}$  holds;

- (iii) there exists a quasi-metric  $\rho$  in  $X$  such that  $d, \rho$  are uniformly locally identical and  $\mathcal{B}$  is the collection of all  $\rho$ -bounded sets.
- (iv) there exists a  $(d, \rho_u)$ -uniformly continuous  $(\tau(d), \tau(d^{-1}))$ -characteristic function for  $\mathcal{B}$ ;

*Proof.* Assume (i) and suppose that  $\rho$  is a uniformly equivalent with  $d$  quasi-metric in  $X$  such that  $\mathcal{B}$  is the collection of all  $\rho$ -bounded sets. Let  $x_0 \in X$  and, for  $n \in \omega$ , let  $B_n = B_\rho(x_0, n+1)$ . We choose  $\delta \in (0; +\infty)$  such that  $\rho(x, y) < \frac{1}{2}$  whenever  $d(x, y) < \delta$ . Then  $[B_n]_d^\delta \subseteq [B_n]_\rho^{\frac{1}{2}} \subseteq B_{n+1}$  for each  $n \in \omega$ , so (ii) follows from (i).

Now, let us suppose that (ii) holds. We may assume that  $\delta \in (0; +\infty)$  and that  $\{B_n : n \in \omega\}$  is a base for  $\mathcal{B}$  such that  $B_0 = \emptyset, B_1 \neq \emptyset$  and  $[B_n]_d^\delta \subseteq B_{n+1}$  for each  $n \in \omega$ . We shall mimic the proof to Proposition 2.2 in [GM] and change parts of it to show that (iii) follows from (ii). We define  $\phi_0(x) = 1$  for each  $x \in X$ . If  $n \in \omega \setminus \{0\}$ , we define  $\phi_n(x) = \min\{1, \frac{1}{\delta}d(B_n, x)\}$  for each  $x \in X$ . It is easy to check that the function  $\phi_n$  is  $(d, \rho_u)$ -uniformly continuous; moreover,  $\phi_n(B_n) \subseteq \{0\}$  and  $\phi_n(X \setminus B_{n+1}) \subseteq \{1\}$ . Let us consider the function  $\chi : X \rightarrow [0; +\infty)$  defined by

$$\chi(x) = n - 2 + \phi_{n-1}(x)$$

for each  $x \in B_n \setminus B_{n-1}$  and for each  $n \in \omega \setminus \{0\}$ . To prove that  $\chi$  is  $(d, \rho_u)$ -uniformly continuous, let us consider an arbitrary pair  $x, y$  of points of  $X$  such that  $d(x, y) < \delta$ . Let  $n \in \omega$  be the unique natural number such that  $x \in B_n \setminus B_{n-1}$ . If  $z \in X \setminus B_{n+1}$ , then  $d(x, z) \geq \delta$ . This implies that  $y \in B_{n+1}$ . Let  $m \in \omega \setminus \{0\}$  be the unique natural number such that  $y \in B_m \setminus B_{m-1}$ . Then  $m \leq n+1$ . We have  $\chi(y) - \chi(x) = m - n + \phi_{m-1}(y) - \phi_{n-1}(x)$ . If  $m = n+1$ , then  $\chi(y) - \chi(x) = 1 + \phi_n(y) - \phi_{n-1}(x) = \phi_n(y) - \phi_n(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta}d(x, y)$ . If  $m = n$ , then  $\chi(y) - \chi(x) = \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{1}{\delta}d(x, y)$ . Suppose that  $m < n$ . Then  $m - n + 1 \leq 0, x \notin B_m, y \in B_{n-1}$  and  $\chi(y) - \chi(x) = m - n + 1 + \phi_{m-1}(y) - \phi_{m-1}(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta}d(x, y)$ . In consequence,  $\chi$  is  $(d, \rho_u)$ -uniformly continuous. Therefore,  $\chi : (X, \tau(d), \tau(d^{-1})) \rightarrow (\mathbb{R}, u, l)$  is bicontinuous. Since, for  $A \subseteq X$ , we have that  $A \in \mathcal{B}$  if and only if  $\chi$  is bounded on  $A$ , the function  $\chi$  is a  $(\tau(d), \tau(d^{-1}))$ -characteristic function for  $\mathcal{B}$ . In much the same way as in Remark 2.5 of [GM], we can define  $\rho(x, y) = \max\{\min\{d(x, y), 1\}, \frac{\delta}{2} \max\{\chi(y) - \chi(x), 0\}\}$  to get a quasi-metric  $\rho$  uniformly locally identical with  $d$  such that  $\mathcal{B}$  is the collection of all  $\rho$ -bounded sets. Thus (ii) implies (iii).

Let us assume that (iii) holds. We take a quasi-metric  $\rho$  in  $X$  such that  $d$  and  $\rho$  are uniformly locally identical and  $\mathcal{B} = \mathcal{B}_\rho(X)$ . We fix  $x_0 \in X$  and define  $f(x) = \rho(x_0, x)$  for  $x \in X$  to get a  $(\tau(\rho), \tau(\rho^{-1}))$ -characteristic function  $f$  for  $\mathcal{B}$  such that  $f$  is  $(d, \rho_u)$ -uniformly continuous. Hence (iii) implies (iv).

Finally, we suppose that (iv) holds and we consider an arbitrary function  $g$  such that  $g$  is a  $(\tau(d), \tau(d^{-1}))$ -characteristic function for  $\mathcal{B}$  and  $g$  is  $(d, \rho_u)$ -uniformly continuous. For  $x, y \in X$ , we can define  $d_g(x, y) = \min\{d(x, y), 1\} + \max\{g(y) - g(x), 0\}$  to see that (iv) implies (i).  $\square$

**Corollary 6.6.** *Theorem 2.4 and Remark 2.5 of [GM] hold true in **ZF**.*

One can use Example 10.16 (i)-(iii) given at the end of Section 10 to see that, for a quasi-metric  $d$  in  $X$  and a bornology  $\mathcal{B}$  in  $X$ , it may happen that the bornological universe  $((X, \tau(d)), \mathcal{B})$  is quasi-metrizable or even metrizable, while  $\mathcal{B}$  is not uniformly quasi-metrizable with respect to  $d$ .

## 7 Applications to independence from **ZF**

Mimicking [GM], let us consider the following bornologies in a metric space  $(X, d)$ : the bornology  $\mathbf{FB}(X)$  of all finite subsets of  $X$ , the bornology  $\mathbf{CB}_d(X)$  generated by the compact subsets of  $(X, d)$ , the bornology  $\mathbf{TB}_d(X)$  of all totally bounded subspaces of  $(X, d)$ , as well as the bornology  $\mathbf{BB}_d(X)$  of all Bourbaki-bounded sets. Several theorems about equivalents of the uniform metrizability of the bornologies  $\mathbf{FB}(X)$ ,  $\mathbf{CB}_d(X)$ ,  $\mathbf{TB}_d(X)$  and  $\mathbf{BB}_d(X)$  in **ZFC** were proved in [GM]. We are going to show that some of the above-mentioned theorems of [GM] are independent of **ZF**, while other theorems of [GM] can be proved in **ZF**. Clearly, we have already shown in the previous section that both Proposition 2.2 and Theorem 2.4 of [GM] hold true in **ZF**.

The following theorem will be helpful:

**Theorem 7.1.** *Equivalent are:*

- (i)  $\mathbf{CC}(fin)$ ;
- (ii) *for every discrete space  $X$ , the bornological universe  $(X, \mathbf{FB}(X))$  is metrizable (in the sense of Hu) if and only if  $X$  is countable.*

*Proof.* Assume (i) and let  $X$  be any discrete space such that the bornological universe  $(X, \mathbf{FB}(X))$  is metrizable. It follows from Theorem 4.7 that  $\mathbf{FB}(X)$  has a countable base. If  $\mathcal{A}$  is a countable base for  $\mathbf{FB}(X)$ , then  $X = \bigcup \mathcal{A}$ , so, by Proposition 3.5 of [Her],  $X$  is countable if  $\mathbf{CC}(\text{fin})$  holds. It is obvious that if  $X$  is a countable discrete space, then the bornological universe  $(X, \mathbf{FB}(X))$  is metrizable in  $\mathbf{ZF}$  by Theorem 4.7

Now, assume that  $\mathbf{CC}(\text{fin})$  fails. Then, in view of Proposition 3.5 of [Her], there exists a sequence  $(A_n)_{n \in \omega}$  of pairwise disjoint non-void finite sets such that the set  $Z = \bigcup_{n \in \omega} A_n$  is uncountable. Let us equip  $Z$  with its discrete topology. Then the collection  $\{\bigcup_{m \in n} A_m : n \in \omega\}$  is a countable base for  $\mathbf{FB}(Z)$ . Then, by Theorem 4.7, the bornological universe  $(Z, \mathbf{FB}(Z))$  is metrizable. This contradicts (ii).  $\square$

For a set  $X$  and a cardinal number  $\kappa$ , let us use the notation  $[X]^{\leq \kappa}$  for the collection of all subsets  $A$  of  $X$  such that  $A$  is of cardinality at most  $\kappa$  and the notation  $[X]^{< \kappa}$  for the collection of all subsets of  $X$  that are of cardinality  $< \kappa$ . (cf. Definition I.13.19 of [Ku2]). Then  $[X]^{< \omega} = \mathbf{FB}(X)$ , while  $[X]^{\leq \omega}$  is the bornology of all at most countable subsets of  $X$ .

The proof to the following interesting theorem is somewhat more complicated than to Theorem 7.1.

**Theorem 7.2.** *Equivalent are:*

- (i) *for every sequence  $(X_n)_{n \in \omega}$  of non-void at most countable sets  $X_n$ , the product  $\prod_{n \in \omega} X_n$  is non-void;*
- (ii) *for every discrete space  $X$ , the bornological universe  $(X, [X]^{\leq \omega})$  is metrizable if and only if  $X$  is countable.*

*Proof.* Assume (i). Let  $X$  be a discrete space such that the bornological universe  $(X, [X]^{\leq \omega})$  is metrizable. Then, by Theorem 4.7, there exists a countable base  $\mathcal{B} = \{X_n : n \in \omega\}$  for the bornology  $[X]^{\leq \omega}$ . Suppose that  $X$  is uncountable. We may assume that  $X_n \subseteq X_{n+1}$  and that  $X_n \neq X_{n+1}$  for each  $n \in \omega$ . By (i), there exists  $x \in \prod_{n \in \omega} (X_{n+1} \setminus X_n)$ . Then, for such an  $x$ , if  $A = \{x(n) : n \in \omega\}$ , then  $A \in [X]^{\leq \omega}$ , while there does not exist  $n \in \omega$  such that  $A \subseteq X_n$ . This is impossible because  $\mathcal{B}$  is a base for  $[X]^{\leq \omega}$ . Therefore, (i) implies (ii).

Now, let us suppose that (i) is false. Consider any sequence  $(X_n)_{n \in \omega}$  of non-empty countable sets such that  $\prod_{n \in \omega} X_n = \emptyset$ . For each  $n \in \omega$ , the set  $Y_n = \prod_{i \leq n+1} X_i$  is countable and non-empty. In much the same way as in

the proof to Theorem 2.12 of [Her], we can show that there does not exist an infinite set  $M \subseteq \omega$  such that  $\prod_{n \in M} Y_n \neq \emptyset$ . Let  $Y = \bigcup_{n \in \omega} Y_n$  and let  $f : \omega \rightarrow Y$  be an injection. Then the set  $M_f = \{n \in \omega : f(\omega) \cap Y_n \neq \emptyset\}$  is finite. This proves that  $Y$  is uncountable and if  $A_n = \bigcup_{m \in n+1} Y_m$  for  $n \in \omega$ , then the collection  $\mathcal{A} = \{A_n : n \in \omega\}$  is a countable base for  $[Y]^{\leq \omega}$ . If we equip  $Y$  with its discrete topology, we will obtain that (ii) is false. Hence (ii) implies (i).  $\square$

*Remark 7.3.* It is unknown to us whether there is a model for **ZF** in which **CUT** fails (cf. [Her]), while condition (i) of Theorem 7.2 is satisfied.

*Remark 7.4.* It is evident that conditions (1) and (2) of Theorem 2.6 of [GM] are equivalent in **ZF**. In view of our Theorem 6.5 and the proof of (3)  $\Rightarrow$  (1) of Theorem 2.6 given in [GM], we have that (3)  $\Rightarrow$  (1) of Theorem 2.6 of [GM] holds true in **ZF**. Since **CC(fin)** is relatively independent of **ZF**, it follows from our Theorem 7.1 that Theorem 2.6 of [GM] is relatively independent of **ZF**. If **M** is a model for **ZF** +  $\neg$ **CC(fin)**, then Theorems 7.1 and 6.5 show that there exists in **M** an uncountable metric space  $(X, d)$  such that **FB**( $X$ ) is uniformly metrizable with respect to  $d$ , so Theorem 2.6 of [GM] fails in **M**. Now, we can deduce from Proposition 3.5 of [Her] that Theorem 2.6 of [GM] is equivalent with **CC(fin)**.

*Remark 7.5.* Let us notice that both (1)  $\Leftrightarrow$  (2) and (3)  $\Rightarrow$  (1) of Theorem 3.1 of [GM] hold true in **ZF**. Unfortunately, Theorem 3.1 of [GM] is relatively independent of **ZF**. Namely, in much the same way as in Remark 7.4, we can show that in every model **M** for **ZF** +  $\neg$ **CC(fin)**, there exists an uncountable set  $X$  such that, for the discrete metric  $d$  in  $X$ , the bornology **CB** $_d(X)$  is uniformly metrizable with respect to  $d$ , while  $(X, d)$  is not Lindelöf but it is obviously uniformly locally compact.

*Remark 7.6.* As Gutierrez showed in [Gut], while working with completions of metric spaces, one must be more careful in **ZF** than in **ZF** + **CC**. Let us observe that if **M** is a model for **ZF** such that there is in **M** an uncountable set  $X$  such that **FB**( $X$ ) is uniformly metrizable with respect to the discrete metric  $d$  in  $X$ , then **TB** $_d(X) = \mathbf{FB}(X) = \mathbf{BB}_d(X)$  is uniformly metrizable, while  $(X, d)$  is neither Lindelöf nor Bourbaki-separable. Therefore, Theorems 4.2 and 5.8 of [GM] fail in **M**. In the light of our Theorem 7.1 and of the fact that **CC(fin)** is relatively independent of **ZF**, Theorems 4.2 and 5.8 of [GM] are relatively independent of **ZF**.

Since many articles about bornologies have been published so far, it may take a lot of time to investigate which of the theorems in the articles can fail in some models for **ZF**. There are theorems about connections between bornologies and realcompactifications that have already appeared in print (cf. [Vr2]) and they seem to be unprovable in **ZF**. In view of Theorem 10.12 of [PW], perhaps, some of them can be proved in **ZF** + **UFT** where **UFT** stands for the Ultrafilter Theorem (cf. [Her]). Let us leave it as an open problem which of the theorems about bornologies that have been proved by other authors in **ZFC** may fail in models for **ZF** and which of them can be proved under weaker assumptions than **ZFC**. We have given only a partial solution to this problem.

## 8 Compact bornologies in quasi-metric spaces

In the light of Remark 7.5, Theorem 3.1 of [GM] may fail in a model for **ZF**. We are going to prove in **ZF** its modified version for compact bornologies in quasi-metric spaces.

For a topological space  $(X, \tau)$ , let  $\mathbf{CB}_\tau(X)$  be the bornology in  $X$  generated by the collection of all compact subsets of  $(X, \tau)$ . If it is useful, we shall use the notation  $\mathbf{CB}((X, \tau))$  for  $\mathbf{CB}_\tau(X)$ .

**Definition 8.1.** Let  $d$  be a quasi-metric in  $X$ .

- (i) We denote by  $\mathbf{CB}_d(X)$  the bornology  $\mathbf{CB}_{\tau(d)}(X)$ .
- (ii) We say that  $X$  is **uniformly locally compact with respect to  $d$**  if there exists  $\delta \in (0; +\infty)$  such that  $B_d(x, \delta) \in \mathbf{CB}_d(X)$  for each  $x \in X$ .

The following example shows that, contrary to compact bornologies in metric spaces, it may happen that, for a quasi-metric  $d$  in  $X$ , there is a set  $A \in \mathbf{CB}_d(X)$  such that  $\text{cl}_{\tau(d)} A \notin \mathbf{CB}_d(X)$ .

**Example 8.2.** Let us consider the set  $X = X_1 \cup X_2$  where  $X_1 = \{\frac{1}{2^{2n}} : n \in \omega\}$  and  $X_2 = \{\frac{1}{2^{2n+1}} : n \in \omega\}$ . Let  $x, y \in X$ . If  $x = y$ , we put  $d(x, y) = 0$ . When  $x \neq y$ , we put  $d(x, y) = 1$  if either  $x, y \in X_1$  or  $x, y \in X_2$ , or  $x \in X_1, y \in X_2$ ; moreover, we put  $d(x, y) = y$  if  $x \in X_2, y \in X_1$ . In this way, we have defined a quasi-metric on  $X$  such that, for each  $y \in X_2$ , the set  $A_y = \{y\} \cup X_1$  is compact in  $(X, \tau(d))$ , while  $\text{cl}_{\tau(d)} A_y = X \notin \mathbf{CB}_d(X)$ .

**Definition 8.3.** We say that a topological space  $(X, \tau)$  is  $\sigma\text{-CB}$  if there exists a countable collection  $\mathcal{A} \subseteq \mathbf{CB}_\tau(X)$  such that  $X = \bigcup \mathcal{A}$ .

*Remark 8.4.* Clearly, it holds true in  $\mathbf{ZF}$  that every  $\sigma$ -compact space is  $\sigma\text{-CB}$  and every  $\sigma\text{-CB}$  Hausdorff space is  $\sigma$ -compact. In every model for  $\mathbf{ZF} + \mathbf{CC}$ , a topological space is  $\sigma$ -compact if and only if it is  $\sigma\text{-CB}$ . We do not know whether there is a model for  $\mathbf{ZF} + \neg\mathbf{CC}$  in which a topological space can be simultaneously  $\sigma\text{-CB}$  and not  $\sigma$ -compact.

**Theorem 8.5.** *Let  $d$  be a (quasi)-metric in  $X$ . Then the following conditions are equivalent:*

- (i)  $\mathbf{CB}_d(X)$  is uniformly (quasi)-metrizable with respect to  $d$ ;
- (ii)  $X$  is uniformly locally compact with respect to  $d$  and  $(X, \tau(d))$  is  $\sigma\text{-CB}$ .

*Proof.* Assume (i). Let  $\rho$  be a uniformly equivalent with  $d$  quasi-metric in  $X$  such that  $\mathbf{CB}_d(X)$  is the collection of all  $\rho$ -bounded sets. There exists  $\delta \in (0; +\infty)$  such that  $\rho(x, y) < 1$  whenever  $d(x, y) < \delta$ . Then, for each  $x \in X$ , we have  $B_d(x, \delta) \subseteq B_\rho(x, 1) \in \mathbf{CB}_d(X)$ , so  $X$  is uniformly locally compact with respect to  $d$ . Moreover, since, by Theorem 4.7,  $\mathbf{CB}_d(X)$  has a countable base, we deduce that  $(X, \tau(d))$  is  $\sigma\text{-CB}$ .

Now, assume (ii). Let  $\delta \in (0; +\infty)$  be such that, for each  $x \in X$ , we have  $B_d(x, \delta) \in \mathbf{CB}_d(X)$ . Let  $C$  be compact in  $(X, \tau(d))$ . It follows from the compactness of  $C$  that there exists a finite set  $K \subseteq C$  such that  $C \subseteq \bigcup_{x \in K} B_d(x, \frac{\delta}{2})$ . Then  $[C]_d^{\frac{\delta}{2}} \subseteq \bigcup_{x \in K} [B_d(x, \frac{\delta}{2})]_d^{\frac{\delta}{2}} \subseteq \bigcup_{x \in K} B_d(x, \delta) \in \mathbf{CB}_d(X)$ . Therefore,  $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$  (cf. proof to 3.1 in [GM]). This implies that  $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$  whenever  $C \in \mathbf{CB}_d(X)$ .

Let  $\mathcal{A} = \{A_n : n \in \omega\} \subseteq \mathbf{CB}_d(X)$  be such that  $X = \bigcup \mathcal{A}$ . We may assume that  $A_n \subseteq A_{n+1}$  for each  $n \in \omega$ . If  $C$  is compact in  $(X, \tau(d))$  then, since  $C \subseteq \bigcup_{n \in \omega} [A_n]_d^{\frac{\delta}{2}}$ , there exists  $m \in \omega$  such that  $C \subseteq \bigcup_{n \in m+1} [A_n]_d^{\frac{\delta}{2}} = [A_m]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$ . Therefore, the collection  $\{[A_n]_d^{\frac{\delta}{2}} : n \in \omega\}$ , is a countable base for  $\mathbf{CB}_d(X)$ . This, together with the fact that  $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$  whenever  $C \in \mathbf{CB}_d(X)$ , implies that there exists a subsequence  $(B_n)_{n \in \omega}$  of the sequence  $([A_n]_d^{\frac{\delta}{2}})_{n \in \omega}$  such that  $[B_n]_d^{\frac{\delta}{2}} \subseteq B_{n+1}$  for each  $n \in \omega$ . Then  $\{B_n : n \in \omega\}$  is a base for  $\mathbf{CB}_d(X)$ . This, together with Theorem 6.5, implies that (i) follows from (ii).  $\square$

**Example 8.6.** Let  $(X, d)$  be the quasi-metric space from Example 8.2. Then condition (ii) of Theorem 8.5 is satisfied; hence, the bornology  $\mathbf{CB}_d(X)$  is uniformly quasi-metrizable with respect to  $d$ . We can also define a uniformly locally identical with  $d$  quasi-metric  $\rho$  in  $X$  such that  $\mathbf{CB}_d(X) = \mathcal{B}_\rho(X)$ . To do this, let us consider  $x, y \in X$ . We put  $\rho(x, y) = 0$  if  $x = y$ . Now, suppose that  $x \neq y$ . Then  $\rho(x, y) = 1$  if  $x, y \in X_1$ . For  $x \in X_2$  and  $y \in X_1$ , we define  $\rho(x, y) = y$ . Finally, for  $x \in X$  and  $y \in X_2$ , we put  $\rho(x, y) = \frac{1}{y}$ .

Using similar arguments to the ones of the proof to Theorem 8.5, we deduce the following corollary:

**Corollary 8.7** (cf. [WJ], Theorem 3.1 of [GM] and Corollary 3.3 of [GM]). *For every metric space  $(X, d)$ , it holds true in  $\mathbf{ZF}$  that  $\mathbf{CB}_d(X)$  is uniformly metrizable with respect to  $d$  if and only if  $(X, d)$  is both  $\sigma$ -compact and uniformly locally compact.*

## 9 Fundamental bornologies in gtses

A new problem is to find an appropriate definition of (quasi)-metrizability for a generalized topological space (in abbreviation: a gts) in the sense of Delfs and Knebusch. Since the notion of a gts in this sense is rather complicated (cf. [DK], [P1]) and it seems that it is still not commonly known to the mathematical community, let us recall it to make our paper more legible.

**Definition 9.1** (cf. Definition 2.2.2 in [P1]). A **generalized topological space** in the sense of Delfs and Knebusch (abbreviated to gts) is a triple  $(X, \text{Op}_X, \text{Cov}_X)$  where  $X$  is a set for which  $\text{Op}_X \subseteq \mathcal{P}(X)$ , while  $\text{Cov}_X \subseteq \mathcal{P}(\text{Op}_X)$  and the following conditions are satisfied:

- (i) if  $\mathcal{U} \subseteq \text{Op}_X$  and  $\mathcal{U}$  is finite, then  $\bigcup \mathcal{U} \in \text{Op}_X, \bigcap \mathcal{U} \in \text{Op}_X$  and  $\mathcal{U} \in \text{Cov}_X$ ;
- (ii) if  $\mathcal{U} \in \text{Cov}_X, V \in \text{Op}_X$  and  $V \subseteq \bigcup \mathcal{U}$ , then  $\{U \cap V : U \in \mathcal{U}\} \in \text{Cov}_X$ ;
- (iii) if  $\mathcal{U} \in \text{Cov}_X$  and, for each  $U \in \mathcal{U}$ , we have  $\mathcal{V}(U) \in \text{Cov}_X$  such that  $\bigcup \mathcal{V}(U) = U$ , then  $\bigcup_{U \in \mathcal{U}} \mathcal{V}(U) \in \text{Cov}_X$ ;
- (iv) if  $\mathcal{U} \subseteq \text{Op}_X$  and  $\mathcal{V} \in \text{Cov}_X$  are such that  $\bigcup \mathcal{V} = \bigcup \mathcal{U}$  and, for each  $V \in \mathcal{V}$  there exists  $U \in \mathcal{U}$  such that  $V \subseteq U$ , then  $\mathcal{U} \in \text{Cov}_X$ ;



- (v) if  $\mathcal{U} \in \text{Cov}_X$ ,  $V \subseteq \bigcup_{U \in \mathcal{U}} U$  and, for each  $U \in \mathcal{U}$ , we have  $V \cap U \in \text{Op}_X$ , then  $V \in \text{Op}_X$ .

*Remark 9.2.* If  $(X, \text{Op}_X, \text{Cov}_X)$  is a gts, then  $\text{Op}_X = \bigcup \text{Cov}_X$  and, therefore, we can identify the gts with the ordered pair  $(X, \text{Cov}_X)$  (cf. [P1], [PW]). If this is not misleading, we shall denote a gts  $(X, \text{Cov}_X)$  by  $X$ .

As far as gtses are concerned, we shall use the terminology of [DK], [P1]-[P2] and [PW].

**Definition 9.3** (cf. [P1]). If  $X = (X, \text{Cov}_X)$  and  $Y = (Y, \text{Cov}_Y)$  are gtses, then:

- (i) a set  $U \subseteq X$  is called **open** in the gts  $X$  if  $U \in \text{Op}_X$ ;
- (ii) the collection  $\text{Cov}_X$  is **the generalized topology** in  $X$ ;
- (iii) an **admissible open family** in the gts  $X$  is a member of  $\text{Cov}_X$ ;
- (iv) a mapping  $f : Y \rightarrow X$  is  $(\text{Cov}_Y, \text{Cov}_X)$ -**strictly continuous** (in abbreviation: strictly continuous) if, for each  $\mathcal{U} \in \text{Cov}_X$ , we have  $\{f^{-1}(U) : U \in \mathcal{U}\} \in \text{Cov}_Y$ .

In this section, let us have a brief look at very natural bornologies in generalized topological spaces. In the next section, we apply the bornologies in gtses to our concepts of (quasi)-metrizability in the category **GTS** of generalized topological spaces and strictly continuous mappings.

**Definition 9.4** (cf. Definitions 2.2.13 and 2.2.25 of [P1]). If  $K$  is a subset of a set  $X$ , then we say that a family  $\mathcal{U} \subseteq \mathcal{P}(X)$  is **essentially finite on  $K$**  if there exists a finite  $\mathcal{V} \subseteq \mathcal{U}$  such that  $K \cap \bigcup \mathcal{U} \subseteq \bigcup \mathcal{V}$ .

**Definition 9.5** (cf. Definition 2.2.25 of [P1]). If  $X = (X, \text{Cov}_X)$  is a gts, then a set  $K \subseteq X$  is called **small in the gts  $X$**  if each family  $\mathcal{U} \in \text{Cov}_X$  is essentially finite on  $K$ .

The collection of all small sets of a gts  $X$  is a bornology in  $X$  (cf. Fact 2.2.30 of [P1]).

**Definition 9.6.** For a gts  $X$ , **the small bornology** of  $X$  is the collection  $\text{Sm}(X)$  of all small sets in  $X$ .

$\mathbf{Sm}(X)$  was denoted by  $\mathbf{Sm}_X$  in [P1] but, since we are inspired by [GM] and we use the notation of [GM], we have replaced  $\mathbf{Sm}_X$  by  $\mathbf{Sm}(X)$  partly for elegance, partly for convenience.

**Definition 9.7** (cf. Definition 3.2 of [PW]). If  $X$  is a gts, we call a set  $A \subseteq X$  **admissibly compact** in  $X$  if, for each  $\mathcal{U} \in \text{Cov}_X$  such that  $A \subseteq \bigcup \mathcal{U}$ , there exists a finite  $\mathcal{V} \subseteq \mathcal{U}$  such that  $A \subseteq \bigcup \mathcal{V}$ .

**Definition 9.8.** For a gts  $X$ , **the admissibly compact bornology** of  $X$  is the collection  $\mathbf{ACB}(X)$  of all subsets of admissibly compact sets of the gts  $X$ .

For a collection  $\mathcal{A}$  of subsets of a set  $X$ , we denote by  $\tau(\mathcal{A})$  the weakest among all topologies in  $X$  that contain  $\mathcal{A}$ . For a gts  $(X, \text{Op}_X, \text{Cov}_X)$ , we call the topological space  $X_{\text{top}} = (X, \tau(\text{Op}_X))$  **the topologization of the gts  $X$**  (cf. [PW]).

**Definition 9.9.** Let  $X$  be a gts. We say that a set  $A$  is **topologically compact** in  $X$  if  $A$  is compact in  $X_{\text{top}}$  (cf. Definition 3.2 of [PW]). The compact bornology  $\mathbf{CB}(X_{\text{top}})$  will be called **the compact bornology of the gts  $X$**  and it will be denoted by  $\mathbf{CB}(X)$ .

**Fact 9.10.** *For every gts  $X$ , the inclusion  $(\mathbf{Sm}(X) \cup \mathbf{CB}(X)) \subseteq \mathbf{ACB}(X)$  holds.*

In general, the collections  $\mathbf{Sm}(X) \cup \mathbf{CB}(X)$  and  $\mathbf{ACB}(X)$  can be distinct and neither  $\mathbf{Sm}(X) \subseteq \mathbf{CB}(X)$  nor  $\mathbf{CB}(X) \subseteq \mathbf{Sm}(X)$ .

**Example 9.11.** For  $X = \mathbb{R} \times \{0, 1\}$ , let  $\text{Op}_X$  be the natural topology in  $X$  inherited from the usual topology of  $\mathbb{R}$  and let  $\text{Cov}_X$  be the collection of all families  $\mathcal{U} \subseteq \text{Op}_X$  such that  $\mathcal{U}$  is essentially finite on  $\mathbb{R} \times \{0\}$ . Then, for  $A = [0; 1] \times \{1\}$  and  $B = \mathbb{R} \times \{0\}$ , we have  $A \in \mathbf{CB}(X) \setminus \mathbf{Sm}(X)$  and  $B \in \mathbf{Sm}(X) \setminus \mathbf{CB}(X)$ , while  $A \cup B \in \mathbf{ACB}(X) \setminus (\mathbf{CB}(X) \cup \mathbf{Sm}(X))$ .

For a set  $X$  and a collection  $\Psi \subseteq \mathcal{P}^2(X)$ , we denote by  $\langle \Psi \rangle_X$  the smallest among generalized topologies in  $X$  that contain  $\Psi$ . If  $\mathcal{A} \subseteq \mathcal{P}(X)$ , let  $\text{EssCount}(\mathcal{A})$  be the collection of all essentially countable subfamilies of  $\mathcal{A}$ . We recall that  $\text{EssFin}(\mathcal{A})$  is the collection of all essentially finite subfamilies of  $\mathcal{A}$  (cf. [P1]-[P2] and [PW]).

**Fact 9.12** (cf. Examples 2.2.35 and 2.2.14(8) of [P1]). *Let  $(X, \tau)$  be a topological space. That  $\text{EssFin}(\tau)$  is a generalized topology in  $X$  is true in  $\mathbf{ZF}$ . That  $\text{EssCount}(\tau)$  is a generalized topology in  $X$  is true in  $\mathbf{ZF} + \mathbf{CC}$ .*

*Remark 9.13.* It is unprovable in  $\mathbf{ZF}$  that, for every topological space  $(X, \tau)$ , the collection  $\text{EssCount}(\tau)$  is a generalized topology in  $X$ . Namely, let  $\mathbf{M}$  be a model for  $\mathbf{ZF} + \neg\mathbf{CC}(\text{fin})$ . In view of the proof to Theorem 7.1, there exists in  $\mathbf{M}$  an uncountable set  $X$  such that  $X$  is a countable union of finite sets. Let  $\tau = \mathcal{P}(X)$ . If  $\text{EssCount}(\tau)$  were a generalized topology in  $X$ , the family of all singletons of  $X$  would belong to  $\text{EssCount}(\tau)$  which is impossible since  $X$  is uncountable.

Let us observe that, for the gts  $X$  from Example 9.11, the admissibly compact bornology of  $X$  is generated by  $\mathbf{CB}(X) \cup \mathbf{Sm}(X)$ . That not every gts may share this property is shown by the following example:

**Example 9.14.** ( $\mathbf{ZF} + \mathbf{CC}$ ) For  $X = \omega_1$ , let  $\text{Op}_X$  be the topology induced by the usual linear order in  $\omega_1$  and let  $\text{Cov}_X = \text{EssCount}(\text{Op}_X)$ . Then  $\mathbf{Sm}(X) = \mathbf{FB}(X) \neq \mathbf{CB}(X) \neq \mathbf{ACB}(X) = \mathcal{P}(X)$ .

In what follows, for sets  $X, Y$  with  $Y \subseteq X$  and for  $\Psi \subseteq \mathcal{P}^2(X)$ , we use the notation  $\Psi \cap_2 Y$  from [P1] for the collection of all families  $\mathcal{U} \cap_1 Y = \{U \cap Y : U \in \mathcal{U}\}$  where  $\mathcal{U} \in \Psi$ . We want to describe  $\langle \Psi \cap_2 Y \rangle_Y$  more precisely in the case when  $\Psi \cap_2 Y \subseteq \text{EssFin}(\mathcal{P}(Y))$ . To do this, we need the concept of a complete ring of sets in  $Y$  that was of frequent use in [PW]. Namely, a **complete ring in  $Y$**  is a collection  $\mathcal{C} \subseteq \mathcal{P}(Y)$  such that  $\emptyset, Y \in \mathcal{C}$ , while  $\mathcal{C}$  is closed under finite unions and under finite intersections. For  $\mathcal{A} \subseteq \mathcal{P}(Y)$ , let  $L_Y[\mathcal{A}]$  be the intersection of all complete rings in  $Y$  that contain  $\mathcal{A}$ .

**Proposition 9.15.** *For a set  $X$ , let  $\Psi \subseteq \mathcal{P}^2(X)$ . Suppose that  $Y \subseteq X$  and that each family from  $\Psi$  is essentially finite on  $Y$ . Then the following conditions are satisfied:*

$$(i) \quad \langle \Psi \cap_2 Y \rangle_Y = \text{EssFin}(L_Y[\bigcup(\Psi \cap_2 Y)]) = \text{EssFin}(\bigcup \langle \Psi \cap_2 Y \rangle_Y) = \text{EssFin}(\bigcup \langle \Psi \rangle_X) \cap_2 Y;$$

(ii) *each family from  $\langle \Psi \rangle_X$  is essentially finite on  $Y$ .*

*Proof.* By applying Proposition 2.2.37 of [P1] to the mapping  $\text{id}_Y : Y \rightarrow X$ , we obtain the inclusion  $\langle \Psi \rangle_X \cap_2 Y \subseteq \langle \Psi \cap_2 Y \rangle_Y$  which, together with (i), implies (ii). To prove (i), let us put  $\mathcal{G}_0 = \langle \Psi \cap_2 Y \rangle_Y$ ,  $\mathcal{G}_1 = \text{EssFin}(L_Y[\bigcup(\Psi \cap_2 Y)])$ .

$Y))$ ,  $\mathcal{G}_2 = \text{EssFin}(\bigcup \mathcal{G}_0)$  and  $\mathcal{G}_3 = \text{EssFin}(\bigcup \langle \Psi \rangle_X) \cap_2 Y$ . Obviously,  $\mathcal{G}_0, \mathcal{G}_1$  and  $\mathcal{G}_2$  are generalized topologies in  $Y$ . By Proposition 2.2.53 of [P1], the collection  $\mathcal{G}_3$  is also a generalized topology in  $Y$ . Since  $\Psi \cap_2 Y \subseteq \mathcal{G}_1$  and  $L_Y[\bigcup (\Psi \cap_2 Y)] \subseteq \bigcup \mathcal{G}_0$ , we have  $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_0$ . It follows from the inclusion  $\langle \Psi \rangle_X \cap_2 Y \subseteq \mathcal{G}_0$  that  $\mathcal{G}_3 \subseteq \mathcal{G}_0$ . Since  $\bigcup (\langle \Psi \rangle_X \cap_2 Y)$  is a complete ring of subsets of  $Y$ , we get  $\mathcal{G}_1 \subseteq \mathcal{G}_3$ . This completes our proof to (i).  $\square$

**Definition 9.16.** If  $X = (X, \text{Op}, \text{Cov})$  is a gts, then:

- (i) **the partial topologization of  $X$**  is the gts  $X_{pt} = (X, (\text{Op})_{pt}, (\text{Cov})_{pt})$  where  $(\text{Op})_{pt} = \tau(\text{Op})$  and  $(\text{Cov})_{pt} = \langle \text{Cov} \cup \text{EssFin}(\tau(\text{Op})) \rangle_X$  (cf. Definition 4.1 of [PW]);
- (ii) the gts  $X$  is called **partially topological** if  $X = X_{pt}$  (cf. Definition 2.2.4 of [P1]);
- (iii)  **$\mathbf{GTS}_{pt}$**  is the category of all partially topological spaces and strictly continuous mappings, while the mapping  $pt : \mathbf{GTS} \rightarrow \mathbf{GTS}_{pt}$  is **the functor of partial topologization** defined by:  $pt(X) = X_{pt}$  for every gts  $X$  and  $pt(f) = f$  for every morphism in  $\mathbf{GTS}$  (cf. [AHS], [P1] and Definition 4.2 of [PW]).

**Proposition 9.17.** *Let  $X$  be a gts. Then  $\mathbf{Sm}(X) = \mathbf{Sm}(X_{pt})$ ,  $\mathbf{CB}(X) = \mathbf{CB}(X_{pt})$  and  $\mathbf{ACB}(X_{pt}) \subseteq \mathbf{ACB}(X)$ .*

*Proof.* The equality  $\mathbf{CB}(X) = \mathbf{CB}(X_{pt})$  and both the inclusions  $\mathbf{Sm}(X_{pt}) \subseteq \mathbf{Sm}(X)$  and  $\mathbf{ACB}(X_{pt}) \subseteq \mathbf{ACB}(X)$  are trivial. Let  $X = (X, \text{Op}_X, \text{Cov}_X)$  and let  $\Psi = \text{Cov}_X \cup \text{EssFin}(\tau(\text{Op}_X))$ . Suppose that  $Y \in \mathbf{Sm}(X)$ . Since each family from  $\Psi$  is essentially finite on  $Y$ , we infer from Proposition 9.15 that  $Y \in \mathbf{Sm}(X_{pt})$ .  $\square$

**Definition 9.18.** A **generalized bornological universe** is an ordered pair  $((X, \text{Op}_X, \text{Cov}_X), \mathcal{B})$  such that  $(X, \text{Op}_X, \text{Cov}_X)$  is a gts, while  $\mathcal{B}$  is a bornology in  $X$ .

**Definition 9.19** (cf. Proposition 2.2.71 of [P1]). Let  $\text{Op}_X$  be a complete ring of subsets of a set  $X$ . Then:

- (i) for a collection  $\mathcal{B} \subseteq \mathcal{P}(X)$ , we define

$$\text{EF}(\text{Op}_X, \mathcal{B}) = \{\mathcal{U} \subseteq \text{Op}_X : \forall A \in \mathcal{B} \{A \cap U : U \in \mathcal{U}\} \in \text{EssFin}(\mathcal{P}(A))\};$$

- (ii) for a topology  $\tau$  in  $X$  and for a bornology  $\mathcal{B}$  in  $X$ , **the gts induced by the bornological universe**  $((X, \tau), \mathcal{B})$  is the triple  $\text{gts}((X, \tau), \mathcal{B}) = (X, \tau, \text{EF}(\tau, \mathcal{B}))$ .

In the light of the proof to Proposition 2.1.31 in [P2], we have the following fact:

**Fact 9.20.** *Suppose that  $((X, \tau), \mathcal{B})$  is a bornological universe such that  $\tau \cap \mathcal{B}$  is a base for  $\mathcal{B}$ . Then  $\text{Sm}((X, \tau, \text{EF}(\tau, \mathcal{B}))) = \mathcal{B}$ .*

**Definition 9.21** (cf. Example 2.1.12 of [P2]). For a (quasi)-metric  $d$  on a set  $X$ , the triple  $(X, \tau(d), \text{EF}(\tau(d), \mathcal{B}_d(X)))$  will be called **the gts induced by the (quasi)-metric  $d$** .

**Fact 9.22** (cf. Example 2.1.12 of [P2]). *If  $d$  is a quasi-metric on a set  $X$ , then  $\text{EF}(\tau(d), \mathcal{B}_d(X))$  is a generalized topology in  $X$  and*

$$\text{Sm}((X, \text{EF}(\tau(d), \mathcal{B}_d(X)))) = \mathcal{B}_d(X).$$

## 10 $\mathcal{B}$ -(quasi)-metrization of gtsets

**Definition 10.1.** Suppose that  $(X, \mathcal{B})$  is a generalized bornological universe. Then we say that the gts  $X$  is  **$\mathcal{B}$ -(quasi)-metrizable** or **(quasi)-metrizable with respect to  $\mathcal{B}$**  if the bornological universe  $(X_{\text{top}}, \mathcal{B})$  is (quasi)-metrizable.

**Definition 10.2.** Let  $X$  be a gts and let  $\mathcal{S}$  be either **CB** or **ACB**, or **Sm**. Then we say that  $X$  is  **$\mathcal{S}$ -(quasi)-metrizable** if  $X$  is (quasi)-metrizable with respect to  $\mathcal{S}(X)$ .

With Proposition 9.17 in hand, we can immediately deduce that the following proposition holds:

**Proposition 10.3.** *Let  $X$  be a gts and let  $\mathcal{S}$  be either **CB** or **Sm**. Then  $X$  is  $\mathcal{S}$ -(quasi)-metrizable if and only if  $X_{pt}$  is  $\mathcal{S}$ -(quasi)-metrizable.*

*Remark 10.4.* If  $X$  is a gts, then the **ACB**-(quasi)-metrizability of  $X_{pt}$  is the (quasi)-metrizability of  $X_{pt}$  with respect to **ACB**( $X_{pt}$ ), while the **ACB**-quasi-metrizability of  $X$  is equivalent to the (quasi)-metrizability of  $X_{pt}$  with respect to **ACB**( $X$ ). We do not know whether the **ACB**-(quasi)-metrizability of  $X$  is equivalent to the **ACB**-(quasi)-metrizability of  $X_{pt}$ .

**Definition 10.5.** A gts  $X = (X, \text{Op}_X, \text{Cov}_X)$  is called:

- (i) **locally small** if there exists  $\mathcal{U} \in \text{Cov}_X$  such that  $\mathcal{U} \subseteq \mathbf{Sm}(X)$  and  $X = \bigcup \mathcal{U}$  (cf. Definition 2.1.1 of [P2]);
- (ii) **weakly locally small** if there exists a collection  $\mathcal{U} \subseteq \text{Op}_X \cap \mathbf{Sm}(X)$  such that  $X = \bigcup \mathcal{U}$ .

Our next theorem says about the form of the partial topologization of an **Sm**-(quasi)-metrizable gts  $X$  when  $X_{pt}$  is locally small.

**Theorem 10.6.** *Suppose that  $X = (X, \text{Op}, \text{Cov})$  is a gts such that its partial topologization  $X_{pt} = (X, \text{Op}_{pt}, \text{Cov}_{pt})$  is locally small. Then the following conditions are equivalent:*

- (i)  $X$  is **Sm**-(quasi)-metrizable;
- (ii)  $X_{pt}$  is induced by some (quasi)-metric  $d$ .

*Proof.* In view of Proposition 9.17, we have  $\mathbf{Sm}(X) = \mathbf{Sm}(X_{pt})$ . In consequence, it is obvious that if  $X_{pt}$  is induced by a (quasi)-metric  $d$ , then  $X$  is **Sm**-(quasi)-metrizable. Assume that  $X$  is **Sm**-(quasi)-metrizable and that  $d$  is a (quasi)-metric on  $X$  such that  $\tau(\text{Op}) = \tau(d)$  and  $\mathbf{Sm}(X_{pt})$  is the collection of all  $d$ -bounded sets. Since  $X_{pt}$  is locally small, it follows from Proposition 2.1.18 of [P2] that  $X_{pt}$  is induced by  $d$ .  $\square$

**Fact 10.7.** *If a gts  $X$  is induced by a (quasi)-metric, then  $X$  is locally small and partially topological.*

**Fact 10.8.** (i) *If  $X$  is a locally small gts, then  $X_{pt}$  is locally small.*

(ii) *If a gts  $X$  is such that  $X_{pt}$  is locally small, then  $X$  is weakly locally small.*

(iii) *A gts  $X$  is weakly locally small if and only if  $X_{pt}$  is weakly locally small.*

We are going to present a pair of weakly locally small but not locally small gtses. For  $\Psi \subseteq \mathcal{P}^2(X)$ , we put  $\Psi_0 = \Psi$  and, for  $n \in \omega$ , assuming that the collection  $\Psi_n \subseteq \mathcal{P}^2(X)$  has been defined, we put  $\Psi_{n+1} = (\Psi_n)^+$  where  $^+$  is the operator described in the proof of Proposition 2.2.37 in [P1]. Then  $\langle \Psi \rangle_X = \bigcup_{n \in \omega} \Psi_n$ . The symbols  $\cup_1, \cap_1, \cup_2, \cap_2$  have the same meaning as in [P1].

**Example 10.9.**  $[\mathbf{ZF} + \mathbf{CC}]$ . Suppose that  $Y$  is an uncountable set. For  $n \in \omega$ , we put  $Y_n = Y \times \{n\}$ . Let  $X = \bigcup_{n \in \omega} Y_n$ ,  $\text{Op}_X = \mathbf{FB}(X) \cup \{X\}$  and  $\text{Cov}_X = \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\})$ . The gts  $X = (X, \text{Op}_X, \text{Cov}_X)$  is weakly locally small and not small. If  $X$  were locally small, then  $Y_0$  would be a subset of a small open set (Fact 2.1.21 in [P2]), so  $Y_0$  would be finite. Hence,  $X$  is not locally small. We have  $\{Y_n : n \in \omega\} \in \text{EF}(\tau(\text{Op}_X), \{Y_n : n \in \omega\})$  and all the sets  $Y_n$  are small and open in  $(X, \text{EF}(\tau(\text{Op}_X), \{Y_n : n \in \omega\}))$ , so the gts  $(X, \text{EF}(\tau(\text{Op}_X), \{Y_n : n \in \omega\}))$  is locally small. We put  $\Psi = \text{Cov}_X \cup \text{EssFin}(\tau(\text{Op}_X))$ . Then  $pt(\text{Cov}_X) = \langle \Psi \rangle_X$  is the generalized topology of  $X_{pt}$ . By Proposition 9.15,  $\langle \Psi \rangle_X \subseteq \text{EF}(\tau(\text{Op}_X), \{Y_n : n \in \omega\})$ . Surprisingly, if  $\mathbf{CC}$  holds, then  $X_{pt}$  is not locally small and, in consequence,  $\langle \Psi \rangle_X \subset \text{EF}(\tau(\text{Op}_X), \{Y_n : n \in \omega\})$ . To prove this, let us assume  $\mathbf{ZF} + \mathbf{CC}$ . It is easy to observe the following facts:

**Fact 1.**  $X \notin [X]^{\leq \omega} \cup_1 \mathbf{Sm}(X)$ .

**Fact 2.** Each  $\Psi_n$  is closed with respect to restriction:  $\Psi_n \cap_2 A \subseteq \Psi_n$  for  $A \subseteq X$ .

For  $\mathcal{W} \subseteq \mathcal{P}(X)$ , let us consider the following property:

**P**( $\mathcal{W}$ ):  $\mathcal{W}$  has an uncountable member and  $\mathcal{W} \subseteq [X]^{\leq \omega} \cup_1 \mathbf{Sm}(X)$ .

For  $n \in \omega$ , let  $T(n)$  be the statement:

$T(n)$ : if  $\mathcal{W} \in \Psi_n$  has **P**( $\mathcal{W}$ ), then  $\mathcal{W}$  is essentially finite on  $X \setminus A$  for some countable  $A \subseteq X$ .

We are going to prove by induction that the following fact holds:

**Fact 3.**  $T(n)$  is true for each  $n \in \omega$ .

*Proof.* Let  $\mathcal{W} \in \Psi_0$  have property **P**( $\mathcal{W}$ ). Then, by Fact 1,  $X \notin \mathcal{W}$ . Thus  $\mathcal{W} \in \text{EssFin}(\tau(\text{Op}_X))$ . Hence  $T(0)$  holds. Suppose that  $T(n)$  is true. The *finiteness*, *stability*, and *regularity* induction steps from the proof of Proposition 2.2.37 in [P1] are obvious.

*Transitivity step.* Let  $\mathcal{W} \in \Psi_{n+1}$  have property **P**( $\mathcal{W}$ ). Suppose that  $\mathcal{U} \in \Psi_n$  and  $\{\mathcal{V}(U) : U \in \mathcal{U}\} \subseteq \Psi_n$  are such that  $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{V}(U)$  and, for each  $U \in \mathcal{U}$ , we have  $U = \bigcup \mathcal{V}(U)$ . Consider any  $U \in \mathcal{U}$ . If every member of  $\mathcal{V}(U)$  is countable, then  $U \in [X]^{\leq \omega}$  because  $\mathbf{CC}$  holds and  $\mathcal{V}(U)$  is essentially countable. Suppose  $\mathcal{V}(U)$  has an uncountable member. Since  $\mathcal{V}(U)$  has property **P**( $\mathcal{V}(U)$ ), it follows from the inductive assumption that there is a countable set  $A(U) \subseteq X$  such that  $\mathcal{V}(U)$  is essentially finite on  $X \setminus A(U)$ . Then  $U \in [X]^{\leq \omega} \cup_1 \mathbf{Sm}(X)$  and  $U$  is uncountable. The above implies that  $\mathcal{U}$  has property **P**( $\mathcal{U}$ ). By the assumption, there is a countable  $A \subseteq X$  such that  $\mathcal{U}$  is essentially finite on  $X \setminus A$ . Let  $\mathcal{U}^* \subseteq \mathcal{U}$  be a finite family

such that  $\bigcup \mathcal{U}^* \setminus A = \bigcup \mathcal{U} \setminus A$ . For each  $U \in \mathcal{U}^*$ , the set  $U$  is countable or  $\mathcal{V}(U)$  is essentially finite on  $U \setminus A(U)$ . This implies that there is a countable  $A(\mathcal{W})$  such that  $\mathcal{W}$  is essentially finite on  $X \setminus A(\mathcal{W})$ .

*Saturation step.* Suppose that there exists  $\mathcal{V} \in \Psi_n$  such that  $\bigcup \mathcal{V} = \bigcup \mathcal{W}$  and, for each  $V \in \mathcal{V}$ , there is  $W(V) \in \mathcal{W}$  such that  $V \subseteq W(V)$ . Since  $\mathcal{W} \subseteq [X]^{\leq \omega} \cup_1 \mathbf{Sm}(X)$ , we have  $\mathcal{V} \subseteq [X]^{\leq \omega} \cup_1 \mathbf{Sm}(X)$ . Since  $\mathcal{W}$  has an uncountable member and  $\mathcal{V}$  is essentially countable, also  $\mathcal{V}$  has an uncountable member and has property  $\mathbf{P}(\mathcal{V})$ . By the inductive assumption, there exists a countable  $A(\mathcal{V})$  such that  $\mathcal{V}$  is essentially finite on  $X \setminus A(\mathcal{V})$ . Then  $\mathcal{W}$  is essentially finite on  $X \setminus A(\mathcal{V})$ , too.  $\square$

Suppose that  $X_{pt}$  is locally small. There exists  $\mathcal{W} \in pt(\text{Cov}_X)$  such that  $\mathcal{W} \subseteq \mathbf{Sm}(X)$  and  $X = \bigcup \mathcal{W}$ . Since  $X$  is uncountable and  $\mathcal{W}$  is essentially countable, at least one member of  $\mathcal{W}$  is uncountable, so  $\mathbf{P}(\mathcal{W})$  holds true. By Fact 3, there exists a countable  $A(\mathcal{W})$  such that  $\mathcal{W}$  is essentially finite on  $X \setminus A(\mathcal{W})$ . Then  $X \setminus A(\mathcal{W}) \in \mathbf{Sm}(X)$ . This is impossible by Fact 1.

The example above is not a solution to the following open problem:

*Problem 10.10.* Is it true in **ZF** that if the partial topologization of a gts  $X$  is locally small, then so is  $X$ ?

**Proposition 10.11.** *Suppose that  $X = (X, \text{Op}_X, \text{Cov}_X)$  is a gts and  $\mathcal{B}$  is a bornology in  $X$ . Then the following conditions are equivalent:*

- (i) *the gts  $X$  is (quasi)-metrizable with respect to  $\mathcal{B}$ ;*
- (ii) *the gts  $(X, \text{EF}(\tau(\text{Op}_X), \mathcal{B}))$  is  $\mathbf{Sm}$ -(quasi)-metrizable and  $\tau(\text{Op}_X) \cap \mathcal{B}$  is a base for  $\mathcal{B}$ .*

*Proof.* Assume that (i) holds. Then, by Theorem 4.7, the collection  $\tau(\text{Op}_X) \cap \mathcal{B}$  is a base for  $\mathcal{B}$ . It follows from Fact 9.20 that  $\mathcal{B} = \mathbf{Sm}((X, \text{EF}(\tau(\text{Op}_X), \mathcal{B})))$ . In consequence, (i) implies (ii). On the other hand, we can use Fact 9.20 with both Definitions 9.19 and 10.1 to infer that (i) follows from (ii).  $\square$

**Definition 10.12.** Suppose that  $(X, \mathcal{B})$  is a generalized bornological universe where  $X = (X, \text{Op}_X, \text{Cov}_X)$ . Let us say that  $X$  is **strongly  $\mathcal{B}$ -(quasi)-metrizable** if there exists a (quasi)-metric  $d$  on  $X$  such that  $\mathcal{B}$  is the collection of all  $d$ -bounded sets and  $\text{Op}_X = L_X[\{B_d(x, r) : x \in X \wedge r \in (0; +\infty)\}]$ .



**Definition 10.13.** A **(quasi)-metric gts** is an ordered pair  $(X, d)$  where  $X = (X, \text{Op}_X, \text{Cov}_X)$  is a gts and  $d$  is a (quasi)-metric in  $X$  such that  $\tau(d) = \tau(\text{Op}_X)$ .

**Definition 10.14.** Suppose that  $(X, d)$  is a (quasi)-metric gts and that  $\mathcal{B}$  is a bornology in  $X$ . We say that  $(X, d)$  is **uniformly  $\mathcal{B}$ -(quasi)-metrizable** or **uniformly (quasi)-metrizable with respect to  $\mathcal{B}$**  if the bornology  $\mathcal{B}$  is uniformly (quasi)-metrizable with respect to  $d$ .

*Remark 10.15.* For a bornology  $\mathcal{B}$  in a gts  $X$ , one can find results in the previous sections that deliver necessary and sufficient conditions for  $X$  to be (quasi)-metrizable with respect to  $\mathcal{B}$  (see Theorems 4.7 and 4.15, as well as Corollaries 4.10 and 4.16) and for a metric gts  $(X, d)$  to be uniformly (quasi)-metrizable with respect to  $\mathcal{B}$  (see Theorems 6.5 and 8.5).

Let us use the real lines described in Definition 1.2 of [PW] as our illuminating examples for the notions of (uniform)  $\mathcal{B}$ -(quasi)-metrizability in the category **GTS**.

**Example 10.16.** Let  $\tau_{nat}$  be the natural topology in  $\mathbb{R}$ . For  $x, y \in \mathbb{R}$ , we put  $d_n(x, y) = |x - y|$ ,  $d_{n,1}(x, y) = \min\{d_n(x, y), 1\}$  and

$$d_n^+(x, y) = d_n(\Phi(x), \Phi(y)) \text{ where } \Phi(x) = \begin{cases} e^x, & x < 0, \\ 1 + x, & x \geq 0. \end{cases}$$

Moreover, we define  $d_{n,1}^+(x, y) = \min\{d_n^+(x, y), 1\}$ . Let us observe that the metrics  $d_n$  and  $d_n^+$  are equivalent but not uniformly equivalent.

- (i) We have  $\mathcal{B}_{d_n}(\mathbb{R}) = \mathbf{CB}_{\tau_{nat}}(\mathbb{R})$  and  $\mathcal{B}_{d_n^+}(\mathbb{R}) = \mathbf{UB}(\mathbb{R})$ . Let us observe that, for a fixed  $\delta \in (0; +\infty)$ , there exists  $n(\delta) \in \omega$  such that if  $C_m = [-m; m]$  for  $m \in \omega$  with  $m > n(\delta)$ , then  $(-\infty; m) \subseteq [C_m]_{d_n^+}^\delta$ . This, together with Theorem 6.5, implies that  $\mathcal{B}_{d_n}(\mathbb{R})$  is not uniformly quasi-metrizable with respect to  $d_n^+$ .
- (ii) For the usual topological real line  $\mathbb{R}_{ut}$  (cf. Definition 1.2(i) of [PW]), we have  $\mathbf{FB} = \mathbf{Sm} \subset \mathbf{CB} = \mathbf{ACB}$  and  $\text{int}_{\tau_{nat}} A = \emptyset$  for each  $A \in \mathbf{Sm}(\mathbb{R}_{ut})$ , so the gts  $\mathbb{R}_{ut}$  is not **Sm**-quasi-metrizable and it is **ACB**-metrizable by  $d_n$ . The metric gtses  $(\mathbb{R}_{ut}, d_n)$  and  $(\mathbb{R}_{ut}, d_{n,1})$  are **ACB**-uniformly metrizable. It follows from (i) that the metric gtses  $(\mathbb{R}_{ut}, d_n^+)$  and  $(\mathbb{R}_{ut}, d_{n,1}^+)$  are not uniformly **ACB**-quasi-metrizable.

- (iii) For the real lines  $\mathbb{R}_{lst}$  and  $\mathbb{R}_{lom}$  (cf. Definition 1.2(iv)-(v) of [PW]), we have  $pt(\mathbb{R}_{lom}) = \mathbb{R}_{lst}$  and  $\mathbf{Sm} = \mathbf{CB} = \mathbf{ACB} = \mathcal{B}_{d_n}(\mathbb{R})$ . The metric gtses  $(\mathbb{R}_{lst}, d_n)$  and  $(\mathbb{R}_{lom}, d_n)$  are both uniformly **Sm**-metrizable; however, none of the metric gtses  $(\mathbb{R}_{lom}, d_n^+)$  and  $(\mathbb{R}_{lst}, d_n^+)$  is uniformly **Sm**-metrizable (see (i)).
- (iv) For the real lines  $\mathbb{R}_{l+om}$  and  $\mathbb{R}_{l+st}$  (cf. Definition 1.2(vii)-(viii) of [PW]), we have  $pt(\mathbb{R}_{l+om}) = \mathbb{R}_{l+st}$  and  $\mathbf{CB} = \mathbf{CB}_{\tau_{nat}}(\mathbb{R}) \subset \mathbf{Sm} = \mathbf{ACB} = \mathcal{B}_{d_n^+}(\mathbb{R})$ . Now, it is obvious that both the metric gtses  $(\mathbb{R}_{l+om}, d_n^+)$  and  $(\mathbb{R}_{l+st}, d_n^+)$  are uniformly **ACB**-metrizable by the metric  $d_n^+$ . The gtses  $\mathbb{R}_{l+om}$  and  $\mathbb{R}_{l+st}$  are **Sm**-metrizable. The metric gtses  $(\mathbb{R}_{l+om}, d_n)$  and  $(\mathbb{R}_{l+st}, d_n)$  are uniformly **Sm**-metrizable and uniformly **ACB**-metrizable by  $d_u(x, y) = d_{n,1}(x, y) + |\max(y, 0) - \max(x, 0)|$ .
- (v) Let us consider the gtses  $\mathbb{R}_{om}, \mathbb{R}_{sлом}, \mathbb{R}_{rom}$  and  $\mathbb{R}_{st}$  (cf. Definition 1.2(ii),(iii), (vi) and (x) of [PW]). We have  $pt(\mathbb{R}_{om}) = pt(\mathbb{R}_{sлом}) = pt(\mathbb{R}_{rom}) = \mathbb{R}_{st}$  and  $\mathbf{CB} \subset \mathbf{Sm} = \mathbf{ACB} = \mathcal{P}(\mathbb{R})$ . The real lines  $\mathbb{R}_{om}, \mathbb{R}_{sлом}, \mathbb{R}_{rom}$  and  $\mathbb{R}_{st}$  are **Sm**-metrizable by the metric  $d_{n,1}$  and they are **CB**-metrizable by the metric  $d_n$ .
- (vi) The gts  $\mathbb{R}_{om}$  (cf. Definition 1.2(ii) of [PW]) is strongly **Sm**-metrizable by  $d_{n,1}$ .

In connection with strong **Sm**-(quasi)-metrizability, let us pose the following open problem:

*Problem 10.17.* Find useful simultaneously necessary and sufficient conditions for a gts to be strongly **Sm**-(quasi)-metrizable.

It might be helpful to have a look at several simple examples of gtses of type  $(X, \text{EF}(\tau, \mathcal{B}))$  and compare them with Proposition 10.11.

**Example 10.18. (Gtses from the Sorgenfrey line.)** Let us use the topologies  $\tau_{S,r}$  and  $\tau_{S,l}$  considered in Example 4.12, as well as the quasi-metrics  $\rho_S$ ,  $\rho_{S,1}$  and  $\rho_L$  defined in Example 4.12.

- (i) The gts  $(\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{CB}_{\tau_{nat}}(\mathbb{R})))$  is **Sm**-quasi-metrizable by the quasi-metric  $\rho_0$  defined as follows:

$$\rho_0(x, y) = \begin{cases} y - x, & x \leq y \\ 1 + x - y, & x > y. \end{cases}$$

- (ii) The gts  $(\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{UB}(\mathbb{R})))$  is **Sm**-quasi-metrizable by  $\rho_S$ , while the gts  $(\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{LB}(\mathbb{R})))$  is **Sm**-quasi-metrizable by  $\rho_L$ .
- (iii) The gts  $(\mathbb{R}, \text{EF}(\tau_{S,r}, \mathcal{P}(\mathbb{R})))$  is **Sm**-quasi-metrizable by  $\rho_{S,1}$ .
- (iv) It follows from Theorem 4.7 that the gtses  $(\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{FB}(\mathbb{R})))$  and  $(\mathbb{R}, \text{EF}(\tau_{nat}, \mathbf{FB}(\mathbb{R})))$  are not **Sm**-quasi-metrizable because  $\tau_{S,r} \cap \mathbf{FB}(\mathbb{R})$  is not a base for  $\mathbf{FB}(\mathbb{R})$ .

**Example 10.19. (Quasi-metric gtses from the Sorgenfrey line.)** We use the same notation as in Example 10.18.

- (i) The quasi-metric gts  $((\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{CB}_{\tau_{nat}}(\mathbb{R}))), \rho_S)$  is uniformly **Sm**-quasi-metrizable by  $\rho_0$ .
- (ii) The quasi-metric gts  $((\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{UB}(\mathbb{R}))), \rho_0)$  is uniformly **Sm**-quasi-metrizable by  $\rho_S$ , while the quasi-metric gts  $((\mathbb{R}, \text{EF}(\tau_{S,r}, \mathbf{LB}(\mathbb{R}))), \rho_0)$  is uniformly **Sm**-quasi-metrizable by  $\rho_L$ ,
- (iii) The quasi-metric gts  $((\mathbb{R}, \text{EF}(\tau_{S,r}, \mathcal{P}(\mathbb{R}))), \rho_0)$  is uniformly **Sm**-quasi-metrizable by  $\min\{\rho_0, 1\}$ .

**Example 10.20.** Let us put  $J = [0; 1] \times \{0\}$  and  $J_q = \{q\} \times [0; 1]$ . For  $S = [0; 1] \cap \mathbb{Q}$ , let  $X = J \cup \bigcup_{q \in S} J_q$ . We consider the collection  $\mathcal{B}$  of all sets  $A \subseteq X$  that have the property: there exists a finite  $S(A) \subseteq S$  such that  $A \subseteq J \cup \bigcup_{q \in S(A)} J_q$ .

- (i) Let  $d_e$  be the Euclidean metric in  $X$ . Then, for each  $A \in \mathcal{B}$ , we have  $\text{int}_{\tau(d_e)} A = \emptyset$ , so, for every topology  $\tau_2$  in  $X$ , the bornology  $\mathcal{B}$  is not  $(\tau(d_e), \tau_2)$ -proper. In consequence, the gts  $(X, \text{EF}(\tau(d_e), \mathcal{B}))$  is not **Sm**-quasi-metrizable.
- (ii) We define another metric  $\rho$  in  $X$  as follows. For  $x, y \in [0; 1]$  and  $q, q' \in S$  with  $q \neq q'$ , we put  $\rho((x, 0), (y, 0)) = |x - y|$ ,  $\rho((q, x), (q, y)) = |x - y|$  and  $\rho((q, x), (q', y)) = x + |q - q'| + y$ . Then, for each  $q \in S$  and for any  $a, b \in [0; 1]$  with  $a < b$ , we have  $\{q\} \times (a; b) = \text{int}_{\tau(\rho)}[\{q\} \times (a; b)] \in \mathcal{B}$ . Since there does not exist  $A \in \mathcal{B}$  such that  $J \subseteq \text{int}_{\tau(\rho)} A$ , we deduce that the gts  $(X, \text{EF}(\tau(\rho), \mathcal{B}))$  is not **Sm**-quasi-metrizable. The space  $(X, \tau(\rho))$  can be called **the comb with its hand  $J$  and teeth  $J_q$ ,  $q \in \mathbb{Q}$**  (compare with Example IV.4.7 of [Kn]).

*Remark 10.21.* One can easily reformulate Theorems 4.7 and 4.15 to get simultaneously necessary and sufficient conditions for a bornological biuniverse to be quasi-pseudometrizable. One can also use quasi-pseudometrics instead of quasi-metrics in Theorem 6.5 to obtain conditions equivalent with the uniform quasi-pseudometrizable of a bornology with respect to a given quasi-pseudometric.

**Example 10.22.** The topological space  $(\mathbb{R}, u)$  is not quasi-metrizable (since it is not  $T_1$ ) but it is quasi-pseudometrizable by  $\rho_u$  (see Section 6).

- (i) The gts  $(\mathbb{R}, \text{EF}(u, \mathbf{UB}(\mathbb{R})))$  is **Sm**-quasi-pseudometrizable by  $\rho_u$ .
- (ii) For the gts  $\mathbb{R}_{ul} = (\mathbb{R}, \text{EF}(u, \mathbf{LB}(\mathbb{R})))$  we have  $\mathbf{Sm}(\mathbb{R}_{ul}) = \mathcal{P}(\mathbb{R})$ . This is why  $\mathbb{R}_{ul}$  is **Sm**-quasi-pseudometrizable by  $\rho_{u,1} = \min\{1, \rho_u\}$ .
- (iii) For the gts  $\mathbb{R}_{ub} = (\mathbb{R}, \text{EF}(u, \mathbf{UB}(\mathbb{R}) \cap \mathbf{LB}(\mathbb{R})))$  we have  $\mathbf{Sm}(\mathbb{R}_{ub}) = \mathbf{UB}(\mathbb{R})$ . This is why  $\mathbb{R}_{ub}$  is **Sm**-quasi-pseudometrizable by  $\rho_u$ .
- (iv) The gts  $\mathbb{R}_{uf} = (\mathbb{R}, \text{EF}(u, \mathbf{FB}(\mathbb{R})))$  is not **LB**( $\mathbb{R}$ )-quasi-pseudometrizable because  $\text{int}_u A = \emptyset$  for each  $A \in \mathbf{LB}(\mathbb{R})$ . Here  $\mathbf{Sm}(\mathbb{R}_{uf})$  is the collection of all sets  $A \in \mathbf{UB}(\mathbb{R})$  such that every non-empty subset of  $A$  has a maximal element. Similarly,  $\mathbb{R}_{uf}$  is not **Sm**-quasi-pseudometrizable. Since  $\mathbf{ACB}(\mathbb{R}_{uf}) = \mathbf{CB}(\mathbb{R}_{uf}) = \mathbf{UB}(\mathbb{R})$ , the gts  $\mathbb{R}_{uf}$  is **ACB**-quasi-pseudometrizable by  $\rho_u$ .

## 11 New topological categories

The table of categories in [AHS], among other categories, says about the category **Top** of topological spaces, the category **BiTop** of bitopological spaces and about the category **Bor** of bornological sets. The categories **GTS**, **GTS**<sub>pt</sub>, **SS** of small generalized topological spaces and **LSS** of locally small generalized topological spaces, as well as **SS**<sub>pt</sub> and **LSS**<sub>pt</sub>, were introduced in [P1] and [P2]. We pointed out in [PW] that, while working with categories and proper classes, a modification of **ZF** is required. We assume a suitably modified version of **ZF** suggested in [PW].

In the light of Proposition 9.17 and Fact 10.8, we can state the following:

**Fact 11.1.** *The functor  $pt$  of partial topologization preserves smallness and local smallness. More precisely:*

(i) *pt* restricted to **SS** maps **SS** onto  $\mathbf{SS}_{pt}$ ;

(ii) *pt* restricted to **LSS** maps **LSS** onto  $\mathbf{LSS}_{pt}$ .

All the categories **Top**, **BiTop**, **GTS**,  $\mathbf{GTS}_{pt}$ , **SS**,  $\mathbf{SS}_{pt}$  and **Bor** are topological constructs (cf. [AHS], [Sal], [P1],[P2], [PW] and [H-N]). Since **Top** and **Bor** are topological constructs, it is obvious that the category **Ubor** of bornological universes (cf. Remark 2.2.70 of [P1]) is a topological construct, too. Let us define several more categories and answer the question whether they are topological constructs.

**Definition 11.2** (cf. 1.2.1 in [H-N]). Let  $\mathcal{B}_X$  be a boundedness in a set  $X$  and let  $\mathcal{B}_Y$  be a boundedness in a set  $Y$ . We say that a mapping  $f : X \rightarrow Y$  is  $(\mathcal{B}_X, \mathcal{B}_Y)$ -**bounded** (in abbreviation: **bounded**) if, for each  $A \in \mathcal{B}_X$ , we have  $f(A) \in \mathcal{B}_Y$ .

**Definition 11.3.** Suppose that  $((X, \tau_1^X, \tau_2^X), \mathcal{B}_X)$  and  $((Y, \tau_1^Y, \tau_2^Y), \mathcal{B}_Y)$  are bornological biuniverses. We say that a mapping  $f : X \rightarrow Y$  is a **bounded bicontinuous mapping** from  $((X, \tau_1^X, \tau_2^X), \mathcal{B}_X)$  to  $((Y, \tau_1^Y, \tau_2^Y), \mathcal{B}_Y)$  if  $f$  is bicontinuous with respect to  $(\tau_1^X, \tau_2^X, \tau_1^Y, \tau_2^Y)$  and  $f$  is  $(\mathcal{B}_X, \mathcal{B}_Y)$ -bounded.

**Definition 11.4.** Suppose that  $((X, \text{Cov}_X), \mathcal{B}_X)$  and  $((Y, \text{Cov}_Y), \mathcal{B}_Y)$  are generalized bornological universes. We say that a mapping  $f : X \rightarrow Y$  is a **bounded strictly continuous mapping** from  $((X, \text{Cov}_X), \mathcal{B}_X)$  to  $((Y, \text{Cov}_Y), \mathcal{B}_Y)$  if  $f$  is both  $(\mathcal{B}_X, \mathcal{B}_Y)$ -bounded and  $(\text{Cov}_X, \text{Cov}_Y)$ -strictly continuous.

**Definition 11.5.** A generalized bornological universe  $((X, \text{Cov}_X), \mathcal{B})$  is called:

- (i) **partially topological** if the gts  $(X, \text{Cov}_X)$  is partially topological;
- (ii) **small** if the gts  $(X, \text{Cov}_X)$  is small.

**Definition 11.6.** We define the following categories:

- (i) **BiUBor** where objects are bornological biuniverses and morphisms are bounded bicontinuous mappings;
- (ii) **GeUBor** where objects are generalized bornological universes and morphisms are bounded strictly continuous mappings;

- (iii) **Ge<sub>pt</sub>UBor** where objects are partially topological generalized bornological universes and morphisms are bounded strictly continuous mappings;
- (iv) **SmUBor** where objects are small generalized bornological universes and morphisms are bounded strictly continuous mappings;
- (v) **Sm<sub>pt</sub>UBor** where objects are partially topological small generalized bornological universes and morphisms are bounded strictly continuous mappings.

**Proposition 11.7.** *The categories defined in 11.6 are all topological constructs.*

*Proof.* To check that, for instance, **Ge<sub>pt</sub>UBor** is a topological construct, we mimic the proof to Theorem 4.4 of [PW]. Namely, let us consider a source  $F = \{f_i : i \in I\}$  of mappings  $f_i : X \rightarrow Y_i$  indexed by a class  $I$  where every  $Y_i$  is a partially topological generalized bornological universe and  $Y_i = ((X_i, \text{Cov}_i), \mathcal{B}_i)$ . Let  $\text{Cov}_X$  be the **GTS**-initial generalized topology for  $F$  in  $X$  (cf. Definition 4.3 of [PW]) and let  $\mathcal{B}_X = \bigcap_{i \in I} \{A \subseteq X : f_i(A) \in \mathcal{B}_i\}$ . For  $X = ((X, \text{Cov}_X), \mathcal{B}_X)$ , let  $X_{pt} = (pt((X, \text{Cov}_X)), \mathcal{B}_X)$ . The canonical morphism  $id : X_{pt} \rightarrow X$  is such that all mappings  $f_i \circ id$  are morphisms in **Ge<sub>pt</sub>UBor**. For any object  $Z$  of **Ge<sub>pt</sub>UBor** and a mapping  $h : Z \rightarrow X_{pt}$ , we can observe that if all  $f_i \circ id \circ h$  with  $i \in I$  are morphisms, then  $id \circ h$  is a morphism of **GTS**, so  $pt(h) = h$  is a morphism of **GTS<sub>pt</sub>**. If all  $f_i \circ id \circ h$  are bounded, then  $pt(h) = h$  is bounded, too. That **BiUBor**, **GeUBor**, **SmUBor** and **Sm<sub>pt</sub>UBor** are topological can be proved by using more or less similar arguments.  $\square$

Some other topological constructs relevant to bornologies or to quasipseudometrics were considered in [CL] and [Vr1].

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